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Department of Physics

THE PROPAGATION, REFLECTION AND GENERATION OF TEMPERATURE WAVES; A TRANSMISSION LINE ANALOGY AND THE REFLECTION AT A CHANGE OF CROSS SECTION IN CIRCULAR CYLINDRICAL RODS.

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ABSTRACT

A discussion is given of the physical and mathematical background of the wave phenomena produced in materials by periodic variation of the temperature or heat flux at a boundary. It is shown how the corresponding mathematical problems separate into a superposition of a static and a dynamic problem, where the former satisfies Laplace's equation and the latter the wave equation with a complex propagation constant but with simpler boundary conditions. In the case of the dynamic problem, a rigorous transmission line description is developed for the principal mode temperature waves travelling on cylindrical rods, where the voltage and current correspond respectively to the principal mode temperature (constant over the cross section) and the total heat flux through a cross section. Obstacles or discontinuities in the rod are then exactly represented by lumped circuits on the transmission line, provided that communication between obstacles or sources takes place only via the principal mode with its considerably lower, although still finite, attenuation compared to the higher modes. It is also necessary that the characteristic impedances of different transmission lines meeting at a junction satisfy a particular condition in order that the junction shows the proper reciprocity properties characteristic of a circuit. This condition is shown to be satisfied by the characteristic impedances resulting from the above definitions of current and voltage. A general theorem restricting the value of any thermal impedance to be resistive and capacitative (i.e. restricted to a single quadrant of the complex impedance plane) is proved and discussed.

Using methods analogous to those developed for the corresponding waveguide problems by J. Schwinger, the lumped circuit representing the change of cross section (and of material) in circular rods is shown to be a simple series impedance and a variational expression is obtained for it in terms of the higher mode amplitudes of the temperature field in the circular junction region. A set of linear equations gives these higher mode amplitudes where the coefficients are Bessel function sums. Numerical procedures for evaluating these Bessel function sums, and in particular the asymptotic forms of these sums, are considered in detail. Numerical values of the impedance are explicitly calculated for a range of cross sections and frequencies but no change of material; but data is tabulated from which the impedance for a change of material as well as a change of cross section may be obtained by a simple additional calculation.

The static heat flow problem down a rod with change of cross section and material is shown to be solved by the same lumped series impedance as the dynamic problem, but evaluated at zero frequency.

Analysis is made of the temperature field of the ring source which corresponds to the heat produced in a thin resistor wrapped around the surface of a rod.

A tabulation of symbols and of mathematical information relating to the numerical calculation is given in the appendices.

1. INTRODUCTION

Recent work on a method of measuring thermal properties of metals which uses non-stationary temperature fields, or, more precisely, which uses temperature waves in cylindrical rods, has given rise to questions of the reflection of such waves at various terminations or discontinuities.[1] This report discusses the calculation of such reflections, as well as the general physical background of temperature waves, and carries through a calculation of the reflection at a change of cross section between two circular cylindrical rods.

In Section 2 the equations of the temperature field $T(\vec{r},t)$ are set up on the basis of explicit physical assumptions, and the characteristic solutions or modes obtained in circular cylindrical uniform rods with free sides, for harmonic time dependence. A table is given of values of diffusivities and other thermal properties at various temperatures.

In Section 3 an analysis is made of the behavior of $T(\vec{r},t)$ with various boundary conditions and heat sources, both stationary and time varying; only the form of $T(\vec{r},t)$ after the transients are over is considered, and for simplicity only the principal mode is used. The (harmonically) time-varying problems are expressed as a superposition of a static problem satisfying Laplace's equation, and a dynamic problem satisfying the wave equation, but with simpler boundary conditions.

In Section 4 the transmission line analogy is developed for the description of the behavior of the principal mode, and equivalent voltages and currents, chosen as analogous respectively to the temperature and the heat flow, are introduced. The effects of discontinuities are rigorously represented as lumped circuits inserted in the transmission lines. A condition on the characteristic impedances is obtained, which must be satisfied if reciprocity is to be shown by the linear relations between currents and voltages at the discontinuity, hence is required for a circuit representation of those linear relations. The transmission line formulas are given for the calculation of the reflection coefficients due to various inserted lumped circuits and terminations of the line. A general discussion and proof are given of the theorem that all thermal impedances lie in the first quadrant of the complex impedance plane, hence are resistive and capacitative. Thermal impedance here is the ratio of the (dynamic, complex) temperature to the (dynamic, complex) heat flow through a cross section, of a principal mode thermal wave, and corresponds to the voltage to current ratio of the analogous transmission line.

In Section 5 the mathematical analysis of the equivalent circuit for a change of both cross section and of material at a junction of two circular cylinders, is set up and carried through. The characteristic impedances are uniquely fixed by the requirements of reciprocity and simplicity of the equivalent circuit. An explicit variational expression for the circuit element (which is just a series element between the transmission lines) is obtained and expressed as a sum of higher mode amplitudes. Linear equations for the higher mode amplitudes are given, with coefficients, $C_{\ell m}$, which are explicit Bessel function sums.

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(1) E. Mendoza - Bull. Am. Phys. Soc. <u>28</u> 18 (Jan. 22, 1953)

D. Waldron and M.A.Herlin - Bull. Am. Phys. Soc. <u>28</u> 23 (April 30, 1953)

In Section 6, the various formulas are collected and put in a form suitable for numerical calculation. In particular, the asymptotic forms of the Bessel function sums for the $C_{\ell m}$ are evaluated and used to simplify the calculation. Calculated values of the circuit element Z and reflection coefficient are given for various selected values of the physical parameters. Numerical data are given permitting simple calculation of Z for all changes of cross section and of material and for all frequencies of practical interest. The general coefficient $C_{\ell m}$ for ℓ not equal to m, is shown to depend simply on $C_{0\ell}$ and C_{0m}, so that the labor of computing the square array of coefficients is much reduced. The functions used by Hahn are explicitly evaluated in terms of the same functions used for the discussion of the asymptotic forms of Bessel function sums.

Section 7 discusses the static problem for the change of cross section, namely the heat flow through the junction when the ends are maintained at fixed (different) temperatures. The solution is shown to follow immediately from the same equivalent circuit representing the junction as in the dynamic problem (but with circuit parameters evaluated at zero frequency).

Section 8 derives the temperature field for the physically realistic ring source (corresponding to a thin electrical resistor around the rod) which produces heat uniformly over the area of a ring on the surface of the rod. The solution is readily found by using the appropriate Green's function in the cylindrical region, and is expressed as a mode expansion. Of particular interest are the phase and amplitude of the principal mode wave, to be respectively compared with the phase of the source and the amplitude of the next higher mode.

Appendix I contains a tabulation of the symbols used in this report and their definitions, listed in order of their first appearance in the body of the report.

Appendix II gives procedures for evaluating a definite integral which appears in the sum formulas for the asymptotic Bessel function series.

Appendix III is a sample numerical calculation of the coefficient C_{01} using the formulas for asymptotic summation.

Appendix IV tabulates various numerical quantities useful in the calculation of the lumped series impedance Z or the reflection coefficient R for the change of cross-section. The numerical values of Z and R for a range of values of the ratio of radii and a range of frequencies (but no change of material) are tabulated.

2. EQUATIONS AND MODES OF THE TEMPERATURE FIELD

The phenomenological equation for heat flux $\vec{Q}$ (joules/cm^2 sec) introduces the thermal conductivity K (joules/cm.sec.deg.)

$$\vec{Q} = -K\vec{\nabla}T \tag{2.1}$$

while conservation of (heat) energy requires

$$-\vec{\nabla}\cdot\vec{Q} = \rho C_g(\partial T/\partial t) \tag{2.2}$$

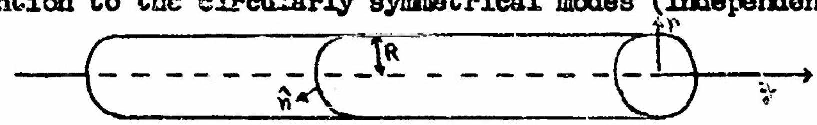
where ρ =density (gms/cm^3), C_g=specific heat (joules/gm.°), T=temperature (deg. Cent.), t=time.(secs.) Putting (2.1) in (2.2) gives the basic differential equation for the temperature field

$$\vec{\nabla}\cdot(K\vec{\nabla}T) = \rho C_g(\partial T/\partial t) \tag{2.3}$$

which will always be treated in the approximation that K is independent of position (except for abrupt discontinuities at junctions of materials which will be considered explicitly), and thus independent of T as well. Then (2.3) takes the form

$$\nabla^2 T = (1/D)(\partial T/\partial t) \quad ; \quad D \equiv K/\rho C_g = \text{diffusivity } (\text{cm}^2/\text{sec}) \tag{2.4}$$

The solutions of (2.4) for $T(\vec{r},t)$ determined by appropriate boundary and initial conditions, are the quantities of interest. In particular the solutions in homogeneous cylindrical rods are the basic mathematical expressions that are needed in the problems to be considered. Hence we give now the "modes" of the temperature field in cylindrical rods for the case of harmonic time dependence. For simplicity, and because the immediate application is to this case, we discuss circular cross sections, although a general theory for arbitrary cross sections can readily be given. Also for these reasons we restrict attention to the circularly symmetrical modes (independent of azimuth).



As usual we introduce the complex harmonically varying temperature

$$T(\vec{r},t) = T^{(0)}(\vec{r})e^{-i\omega t} \tag{2.5}$$

(where $\vec{r}$ is a general position vector and the true physical temperature is either the real or the imaginary part of $T(\vec{r},t)$), and introducing cylindrical coordinates r and z (2.4) becomes

$$\frac{\partial^2 T^{(0)}}{\partial z^2} + \frac{\partial^2 T^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial T^{(0)}}{\partial r} + \frac{i\omega}{D}T^{(0)} = 0 \tag{2.6}$$

The possible solutions of (2.6) are restricted by the boundary condition on the cylindrical surface, at radius r=R. Neglecting radiation, a very good approximation

for the low temperatures (a few degrees Kelvin) in mind, this condition is that $Q\cdot\hat{n}$ =heat flow normal to the surface=0, hence from (2.1)

$$\frac{\partial T}{\partial n} = \left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \; . \tag{2.7}$$

(2.7) will then restrict the possible radial mode functions.

Separation of variables by introducing a product function for $T^{(1)}(\vec{r}) = \varphi(r)\psi(z)$ leads immediately to

$$T^{(1)}(\vec{r}) = \varphi(r) e^{\pm i \varkappa_n z} \tag{2.8}$$

$$\varkappa_n^2 = \varkappa^2 - \gamma_n^2 \quad ; \quad \varkappa^2 = \frac{i\omega}{D} \tag{2.9}$$

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \gamma_n^2 \varphi = 0 \quad ; \quad \left(\frac{\partial \varphi}{\partial r}\right)_{r=R} = 0 \tag{2.10}$$

The solutions of (2.10) are the Bessel functions of zero order, $J_0(\gamma_n r)$, $N_0(\gamma_n r)$. The latter are excluded by the requirement of finiteness at r=0, and the γ_n are then fixed by the boundary condition at r=R giving

$$J_1(\gamma_n R) = 0$$

$$\gamma_n R = \beta_n \quad ; \quad \beta_0 = 0 \, , \; \beta_1 = 3.8317 \, , \; \beta_2 = 7.0156 \, , \; ----$$

$$\beta_n \sim \pi\left(n + \frac{1}{4} - \frac{0.15198}{n+\frac{1}{4}} + \cdots\right) \tag{2.11}$$

(The β_n are tabulated in Appendix IV)
Thus the temperature modes are

$$T(\vec{r},t) = J_0(\gamma_n r)\, e^{\pm i \varkappa_n z - i\omega t} \tag{2.12}$$

where $J_1(\gamma_n R) = 0$, $\varkappa_n^2 = \varkappa^2 - \gamma_n^2$, $n = 0, 1, 2, ---$

The orthogonality and completeness of these modes will be introduced later, but just now we note that in the z direction they take the form of propagating

(and attenuating) waves. In particular, the lowest mode is

$$T(\vec{r},t) = e^{\pm i\varkappa z - i\omega t} = e^{\mp\sqrt{\frac{\omega}{2D}}z}\, e^{\pm i\sqrt{\frac{\omega}{2D}}z - i\omega t} \quad ; \quad \varkappa = (1+i)\sqrt{\frac{\omega}{2D}} \equiv (1+i)k \tag{2.13}$$

Note that even the lowest mode attenuates, although less strongly than higher modes, and that it is uniform over the cross section, thus obviously satisfying the boundary condition. In the following it will be called the principal mode and it will in fact have very much less attenuation than the next higher mode and all others. Thus its attenuation is a pure constant of the material, whereas for all higher modes the attenuation is increased by a geometrical constant, γ_n, arising essentially from the difficulty in fitting a long (radial) wavelength into a narrow tube, and exactly analogous to the "cutoff" condition for electromagnetic waves. It is the wave form of the modes in the z coordinate that gives rise to the transmission line analogy developed in the next section.

Finally we note the static forms of the temperature modes, obtained by letting ω, hence also $\varkappa$, approach 0. (2.6) becomes Laplace's equation and (2.12) (2.13) take the form

$$T(\vec{r}) = Az + B \qquad \text{(principal static mode)}$$
$$= J_0(\gamma_n r)\, e^{\pm\gamma_n z}, \quad n = 1, 2, \cdots \qquad \text{(higher static modes)} \tag{2.14}$$

Table of Thermal Properties

On the following pages are tabulated values of the diffusivity, thermal conductivity, and specific heat (per unit volume) of a number of metals and some other materials at liquid helium, liquid nitrogen and room temperature. These will give some idea of the magnitudes of the physical quantities in the above discussion and, particularly of the diffusivity for various materials as a function of temperature.

The wavelengths λ of temperature waves follow directly from the values of diffusivity and frequency. Since these wavelengths are the basic measured quantities and essentially determine the behavior of the temperature field, they are also tabulated at a frequency of 100 cycles/sec with a conversion table for other frequencies. Since the amplitude of a temperature wave drops by a factor of $e^{-2\pi} = .00188$ or about 0.2% per wavelength the useful length for measurement is less than $(\lambda/2)$ while isolation of two points from each other is effectively achieved at about a wavelength. Hence λ must be neither too long nor too short for convenient measurement.

At room temperature and liquid nitrogen temperature λ is a fraction of a centimetre, which is uncomfortably small for measurement. However, due to the great increase of D at helium temperatures (associated with the decrease in heat capacity below the Debye temperature), λ_{100} has the convenient size of a few centimetres. In a few cases of very pure materials λ_{100} is somewhat too large, (e.g. Sn, Cu, Mg, Zn) but increasing the frequency can easily bring these down to a suitable range. Thus wavelength considerations restrict the temperature wave method to being primarily a low temperature technique at present, not to speak of other technical difficulties connected with temperature measurement and behavior of transients at higher temperatures.

Values of Diffusivities and Other Thermal Properties at Helium, Nitrogen (approx.) and Room Temperatures.*

K = thermal conductivity, given in watts/cm deg

C = heat capacity per unit volume, given in joules/cm^3 deg

= ρC_g , ρ = density (gm/cm^3) ; C_g = specific heat (joules/gm deg)

D = diffusivity , given in (cm^2/sec)

λ_ν = wavelength of temperature wave at frequency ν (given at ν = 100 cycles/sec)

$$= \sqrt{\frac{4\pi D}{\nu}} , \quad \frac{\lambda_\nu}{\lambda_{100}} = \sqrt{\frac{100}{\nu}} ,$$

ν =	1	10	100	500	1000	2000	5000
$\lambda_\nu/\lambda_{100}$ =	10	3.16	1.00	.447	.316	.224	.141

	T (°K)	K (watts / cm deg)	C (joules / cm^3 deg)	D (cm^2 / sec)	λ_{100} (cm)	Purity of sample for measurements of K
Al	2	0.75	3.18 x 10^{-4}	0.236 x 10^4	17.2	99.994 %
	3	1.25	5.03 x 10^{-4}	0.249 x 10^4	17.7	"
	4	1.6	7.54 x 10^{-4}	0.212 x 10^4	16.3	"
	118	2.14	1.55	1.39	0.42	
	292	2.00	2.42	0.827	0.32	
Sb	273	0.183	1.37	0.134	0.13	
Bi	90	0.104	1.07	9.72 x 10^{-2}	0.11	
	294	0.079	1.19	6.64 x 10^{-2}	0.091	
Cd	90	0.997	1.50	0.665	0.29	
	296	0.926	2.00	0.464	0.24	
Graphite	100	0.2	.314	0.637	0.28	
Diamond	100	27	7.34 x 10^{-2}	3.68 x 10^2	6.8	
	260	10	1.77	5.64	0.84	
Cr	112	2.39	1.78	1.34	0.41	
	296	5.58	3.26	1.71	0.46	
Co	2	0.015	0.152 x 10^{-2}	9.87	1.1	99.99 %
	3	0.02	0.24 x 10^{-2}	8.33	1.0	"
	4	0.03	0.316 x 10^{-2}	9.50	1.1	"

	T (°K)	K $\left(\frac{\text{watts}}{\text{cm deg}}\right)$	C $\left(\frac{\text{joules}}{\text{cm}^3\text{deg}}\right)$	D $\left(\frac{\text{cm}^2}{\text{sec}}\right)$	λ 100 (cm)	Purity of sample for measurement of K
Cu	3	11	5.46×10^{-4}	2.01×10^{4}	50	.0005 % Ag
	4	14	9.40×10^{-4}	1.49×10^{4}	43	.0003 % Ni
	118	4.57	2.52	1.82	0.48	
	292	3.83	3.42	1.11	0.37	
Au	294	2.92	2.48	1.17	0.38	
In (normal)	2	1.45	10.63×10^{-4}	0.136×10^{4}	13	99.9 %
	3	2.20	0.292×10^{-2}	7.53×10^{2}	9.7	"
	4	2.75	0.665×10^{-2}	4.14×10^{2}	7.2	"
In (supercon) ducting	2	0.75	10.63×10^{-4}	7.05×10^{2}	9.4	"
	3	2.05	0.346×10^{-2}	5.93×10^{2}	8.6	"
In (normal)	2	5.5	10.63×10^{-4}	0.518×10^{4}	25	99.993 %
	3	7.5	0.292×10^{-2}	0.257×10^{4}	18	"
	4	8.0	0.665×10^{-2}	0.121×10^{4}	12	"
In (supercon-) ducting	2	4.0	10.63×10^{-4}	0.377×10^{4}	22	"
	3	7.5	0.346×10^{-2}	0.218×10^{4}	17	"
Ir	290	0.588	2.65	0.222	0.15	
Fe	2	0.2	0.141×10^{-2}	1.42×10^{2}	4.2	99.99 %
	3	0.35	0.218×10^{-2}	1.61×10^{2}	4.5	"
	4	0.5	0.300×10^{-2}	1.67×10^{2}	4.6	"
	294	0.671	3.51	0.191	0.15	
Pb (normal)	2	13	11.69×10^{-4}	1.12×10^{4}	37	
	3	19	41.2×10^{-4}	0.461×10^{4}	24	
	4	17	84.7×10^{-4}	0.201×10^{4}	16	
	118	0.384	1.32	0.292	0.19	
	292	0.346	1.45	0.238	0.17	
Pb (supercon-) ducting	2	1.0	0.169×10^{-2}	5.92×10^{2}	8.6	
	3	3.5	0.481×10^{-2}	7.27×10^{2}	9.6	
	4	3.0	1.10×10^{-2}	2.73×10^{2}	5.8	
Mg	2	3.0	2.215×10^{-4}	1.36×10^{4}	41	99.95 %
	3	3.5	3.89×10^{-4}	0.90×10^{4}	34	"
	4	4.0	5.99×10^{-4}	0.67×10^{4}	29	"
	293	[illegible]	1.67	0.94	0.34	

	T (°K)	K $\left(\frac{\text{watts}}{\text{cm deg}}\right)$	C $\left(\frac{\text{joules}}{{}^{\circ}\text{cm}^3}\right)$	D $\left(\frac{\text{cm}^2}{\text{sec}}\right)$	λ_{100} (cm)	Purity of sample for measurement of K
Hg (normal)	4	2.0	0.0566	35.4	2.1	
	291	0.821	1.88	0.436	0.23	
Hg (supercon-ducting)	4	1.6	0.0566	28.2	1.9	
Mo	294	1.43	2.49	0.574	0.27	
Ni	2	0.5	0.228×10^{-2}	2.19×10^{2}	5.2	99.997 %
	3	0.8	0.349×10^{-2}	2.30×10^{2}	5.4	"
	4	1.1	0.475×10^{-2}	2.32×10^{2}	5.4	"
	120	0.538	2.42	0.222	0.17	
	294	0.592	3.95	0.150	0.14	
Pd	2	0.2	0.314×10^{-2}	63.6	2.8	99.995 %
	3	0.3	0.477×10^{-2}	62.9	2.8	"
	4	0.4	0.62×10^{-2}	64.5	2.8	"
	294	0.701	2.96	0.237	0.17	
Pt	2	4.0	0.16×10^{-2}	25.1×10^{2}	18	99.999 %
	3	7.0	0.265×10^{-2}	26.3×10^{2}	18	"
	4	9.0	0.403×10^{-2}	22.4×10^{2}	17	"
	294	0.694	2.90	0.239	0.17	
Rh	291	0.875	3.03	0.289	0.19	
Ag	2	1.0	0.0256×10^{-2}	39.0×10^{2}	22	99.99 %
	3	1.5	0.0652×10^{-2}	23.0×10^{2}	17	"
	4	2.0	0.122×10^{-2}	16.4×10^{2}	14	"
	118	4.16	2.03	2.05	0.51	
	292	4.41	2.45	1.80	0.47	
Na	4	~ 9.5	0.248×10^{-2}	38×10^{2}	22	99.9 %
Ta (normal)	2	0.45	0.119×10^{-2}	3.78×10^{2}	6.9	99.9 %
	3	0.75	0.192×10^{-2}	3.91×10^{2}	7.0	"
	4	1.1	0.303×10^{-2}	3.64×10^{2}	6.8	"
	291	0.541	2.50	0.217	0.16	
Ta (supercon-ducting)	2	0.20	0.1034×10^{-2}	1.93×10^{2}	4.9	99.9 %
	3	0.65	0.288×10^{-2}	2.26×10^{2}	5.3	"
	4	1.1	0.613×10^{-2}	1.79×10^{2}	4.7	"
Ta (normal)	3	0.11	0.192×10^{-2}	57.4	2.7	99.98 %
	4	0.14	0.303×10^{-2}	46.2	2.4	"
Ta (supercon-ducting)	3	0.075	0.288×10^{-2}	26.0	1.8	"
	4	0.13	0.613×10^{-2}	21.2	1.6	"

	T (°K)	K $\left(\frac{\text{watts}}{\text{cm deg}}\right)$	C $\left(\frac{\text{joules}}{{}^{\circ}\text{cm}^3}\right)$	D $\left(\frac{\text{cm}^2}{\text{sec}}\right)$	λ (cm)	Purity of sample for measurement of K
Sn (normal) white	2	36	3.24×10^{-4}	11.1×10^4	118	99.996 %
	3	51	7.40×10^{-4}	6.89×10^4	93	"
	4	55	15.5×10^{-4}	3.56×10^4	67	"
	118	0.799	1.23	0.649	0.28	
	291	0.645	1.48	0.435	0.23	
Sn (supercon-) ducting	2	22	3.24×10^{-4}	6.79×10^4	92	"
	3	46	9.93×10^{-4}	4.64×10^4	76	"
	4	55	15.5×10^{-4}	3.56×10^4	67	"
Ti	2	0.015	0.0669×10^{-2}	22.5	1.7	99.99 %
	3	0.02	0.118×10^{-2}	17.0	1.5	"
	4	0.025	0.181×10^{-2}	13.8	1.3	"
Ti	2	0.01	0.0669×10^{-2}	15.0	1.4	99.9 %
	3	0.015	0.118×10^{-2}	12.7	1.3	"
	4	0.02	0.181×10^{-2}	11.1	1.2	"
W	292	1.46	2.74	5.32	0.82	
Zn	2	5	1.96×10^{-4}	2.54×10^4	56	99.9995% Poly-crystalline
	3	7.5	4.15×10^{-4}	1.81×10^4	48	"
	4	10	7.76×10^{-4}	1.29×10^4	40	"
	118	1.16	2.21	0.524	0.26	
	292	1.10	2.77	0.397	0.22	
Zn	2	2.0	1.96×10^{-4}	1.02×10^4	36	99.997% Single crystal
	3	4.0	4.15×10^{-4}	0.963×10^4	33	"
	4	6.0	7.76×10^{-4}	0.772×10^4	31	"
Zr	2	0.02	0.047×10^{-2}	42.6	2.3	~98.0 %
	3	0.03	0.0822×10^{-2}	36.6	2.1	"
	4	0.04	0.126×10^{-2}	31.6	2.0	"
KCl	90	0.434	1.343	0.323	0.249	
He	(λpt)2.186	17.2	2.06	8.35	1.02	Assumes same K as for HeI
	3	20.9	.38	55	2.63	
	4	26.0	.50	52	2.56	

* Data on heat capacities from C. A. Shiffman "The Heat Capacities of the Elements Below Room Temperature", G. E. Report, October, 1952.

Data on helium temperature thermal conductivities from references given in J. Olsen and H. Rosenberg "Thermal Conductivity of Metals at Low Temperatures", Advances in Physics 2, 28 (1953).

Other data from the Handbook of Chemistry and Physics.

3. BOUNDARY CONDITIONS, HEAT SOURCES, AND SEPARATION OF NON-STATIONARY PROBLEMS INTO STATIC AND DYNAMIC PARTS

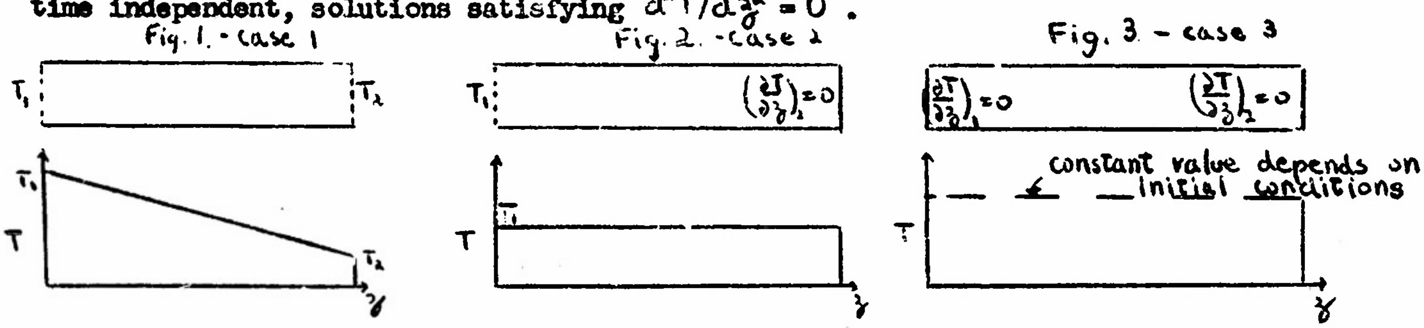
(3a) Steady State Temperature Distributions for Static Boundary Conditions and Heat Sources.

The overall purpose of Section 3 is to show how the general heat conduction problem with time varying boundary conditions, separates mathematically into a static and a dynamic problem for each of the various boundary conditions. Although the measurements are primarily concerned with the dynamic problem, this analysis is necessary because the two problems are usually mixed in the physical situation. Also the nature of the solutions is not always obvious, since in some cases there is no true static solution, independent everywhere of time. The dynamic problem never has such difficulties, however.

Accordingly we start in Subsection (3a) with a tabulation of solutions of some simple static problems with various boundary conditions. These will be useful in discussing the more complex time varying problems, and will develop some feeling for the physical behavior in the situations of interest.

Two types of physical boundary conditions commonly occur, and we illustrate their effects on the temperature field by typical simple (one dimensional) situations. Either T may be fixed over a given surface, say by means of a reservoir (of infinite heat capacity and heat conductivity), or $(\partial T/\partial n)$ may be fixed over the surface. (A still more general boundary condition fixes the ratio of T to $(\partial T/\partial n)$. This is, in fact, the radiation condition unto a vacuum, and, as we shall see below, amounts to specification of the "thermal impedance" of the boundary. Some consideration is given to this boundary condition later, but the possible physical situations are adequately illustrated by the special cases above.) Two possibilities for $(\partial T/\partial n)$ are conveniently distinguished; either $(\partial T/\partial n) = 0$, corresponding to an isolated or free surface across which no heat flows, since by (2.1) $\vec{Q}\cdot n = 0$; or $(\partial T/\partial n)$ is finite and heat does cross the surface. The latter situation corresponds to a source; it will be treated by explicitly introducing the source, and in fact generalized to the case of an internal source. (A third type of termination which may nominally be considered a boundary condition, is infinite extension of the rod in some direction. True static solutions for this case do not exist, but for fixed z after long enough time, static behavior does set in.)

Three combinations of the boundary conditions T fixed or $(\partial T/\partial n) = 0$ are illustrated in the figures for a section of uniform rod. These are all static, i.e. time independent, solutions satisfying $d^2T/dz^2 = 0$.



In Fig. 1 the ends are at fixed values T_1 and T_2, $T_1 > T_2$, and the steady state is $T = T_1 - (z/\ell)(T_1 - T_2)$

In Fig. 2 one end is at T_1 and the other end is free, and the final steady state is $T = T_1$.

In Fig. 3 both ends are free, and any constant value of T is a solution.

(In the diagrams the representation adopted will use solid lines for free walls, and

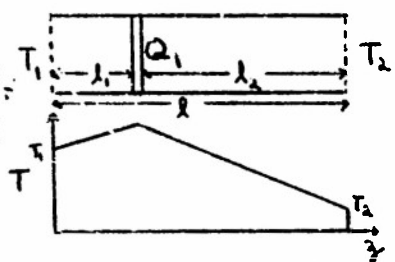
dotted lines for constant temperature surfaces, a choice that seems natural in the sense that free walls are adiabatic, whereas heat passes across the constant T surfaces. The logical extension of this notation will use a double solid line for a heat source, which corresponds to a fixed value of $(\partial T/\partial n)$, or of $\Delta(\partial T/\partial n)$, the discontinuity in $(\partial T/\partial n)$, for an internal source, a wavy solid line for a time varying source etc. These extensions will be illustrated later.)

To complete the tabulation of static problems, consider the steady state temperature distributions when a source, Q_1, is introduced at z_1 into the rod (of cross section area A. For this purpose we use a simplified source which introduces heat uniformly over the cross section (i.e. a circular disc source of zero thickness.) With such a source, T remains uniform over the cross section, but has a discontinuous derivative at z_1 given by:

$$K\left[\left(\frac{\partial T}{\partial z}\right)_{z_1-0}-\left(\frac{\partial T}{\partial z}\right)_{z_1+0}\right]=\frac{Q_1}{A}=-K\,\Delta\left(\frac{\partial T}{\partial z_1}\right) \tag{3.1}$$

The three previous cases become those in Figs. 4,5,6. Thus in Fig. 4 T has two linear sections which intersect at Q_1 and are explicitl given by (3.2).

Fig. 4: Case 1

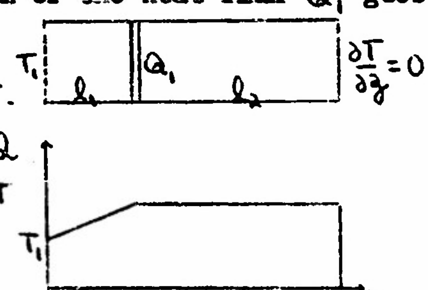


$$T=\frac{T_1(l-z)}{l}+\frac{T_2 z}{l}+\frac{Q_1}{KA}\frac{l_2}{l}z\ ;\ 0\le z\le l_1$$

$$T=\frac{T_1(l-z)}{l}+\frac{T_2 z}{l}+\frac{Q_1}{KA}\frac{l_1}{l}(l-z)\ ;\ l_1\le z\le l_2 \tag{3.2}$$

In Fig. 5, the steady state again has two linear sections intersecting at Q_1 but now all of the heat flux Q_1 goes to the end at fixed T.

Fig. 5- Case 2

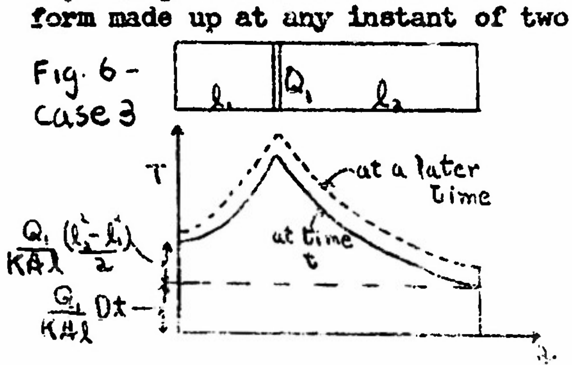


$$T=T_1+\frac{Q_1}{KA}z\ ;\ 0\le z\le l_1$$

$$T=T_1+\frac{Q_1 l}{KA}\ ;\ l_1\le z\le l_2 \tag{3.3}$$

Finally, in the situation in Fig. 6 no true steady state exists since there is no way to dispose of the heat without heating up the rod. However, the transients depending on initial conditions do disappear and the solution approaches a simple form made up at any instant of two parabolic sections intersecting at Q_1 .

Fig. 6 - case 3

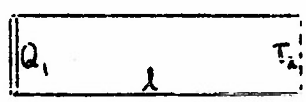


$$T=\frac{Q_1}{KAl}\left[Dt+\frac{z^2}{2}+\frac{l_2^2-l_1^2}{2}+C\right]\ ;\ 0\le z\le l_1$$

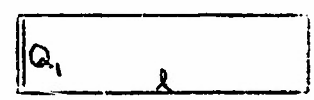
$$T=\frac{Q_1}{KAl}\left[Dt+\frac{(l-z)^2}{2}+C\right]\ ;\ l_1\le z\le l \tag{3.4}$$

As time passes all points of the rod increase their T at the same rate, so that this situation might be characterized as a "quasi-static" state. It is not time independent as in previous cases, but at least is changing uniformly (not periodically) with time, and is independent of the initial conditions (except for the constant C).

Note that (3.3) and (3.4) provide the solutions to the two problems with a source at the boundary, since the source may move to the free end without affecting the solution, e.g.



$$T = T_2 + \frac{Q_1}{KA}(l - z)$$



$$T = \frac{Q_1}{KAl}\left(Dt + \frac{(l-z)^2}{2}\right) \qquad (3.5)$$

The various problems with semi-infinite termination of the rods are solved later.

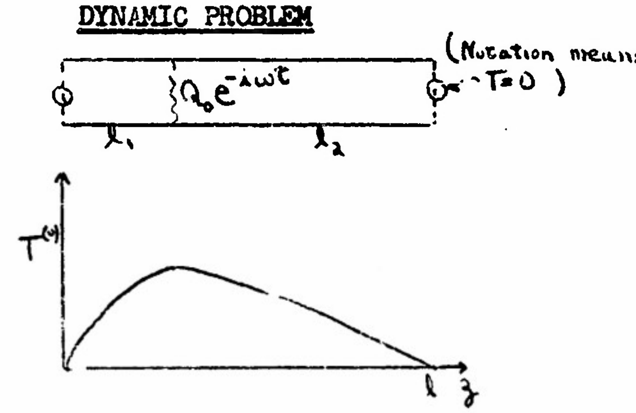
(3b) <u>Periodic Temperature Distribution for Harmonically Varying Sources and Boundary Conditions.</u>

Now suppose the source has a harmonic component so that the heat input is $Q_1 + Q_0 e^{-i\omega t}$. The solution is obtained as the superposition of solutions of two problems which may be designated as the static and the dynamic problems. The static problem has the steady state solution set up by Q_1 and already given. The dynamic problem requires solution of a pure wave propagation problem set up by a source $Q_0 e^{-i\omega t}$ but with the simpler boundary condition $T = 0$ on any surface held at fixed T. This has a solution with harmonic time variation (after the transient has died out), which is independent of the initial conditions, and is conveniently call the periodic state (a suggestion due to E. Mendoza).

We may represent these problems diagrammatically, as follows, with the boundary condition of the three cases corresponding to the constant source problems above.

STATIC PROBLEM (case 1) DYNAMIC PROBLEM

$$\nabla^2 T^{(1)} = 0$$



$$\nabla^2 T^{(0)} + \kappa^2 T^{(0)} = 0 \; ; \; \kappa^2 = i\omega/D$$

Static Problem (cont.)

$$T^{(1)} = T_1 \quad \text{at } z = 0$$
$$\phantom{T^{(1)}} = T_2 \quad \text{at } z = l$$
$$-\Delta\left(\frac{\partial T^{(1)}}{\partial z}\right)_{z=l_1} = \frac{Q_1}{KA}$$
$$\frac{\partial T^{(1)}}{\partial n} = 0$$

Dynamic Problem (cont.)

$$T^{(0)} = 0 \quad \text{at } z = 0, l$$
$$-\Delta\left(\frac{\partial T^{(0)}}{\partial z}\right)_{z=l_1} = \frac{Q_0}{KA}$$
$$\frac{\partial T^{(0)}}{\partial n} = 0$$

The general solution of Case (1) now has the form

$$T = T^{(1)} + T^{(0)} e^{-i\omega t}$$

where $T^{(1)}$ and $T^{(0)}$ are solutions of the static and dynamic problems (3.6) and have the forms

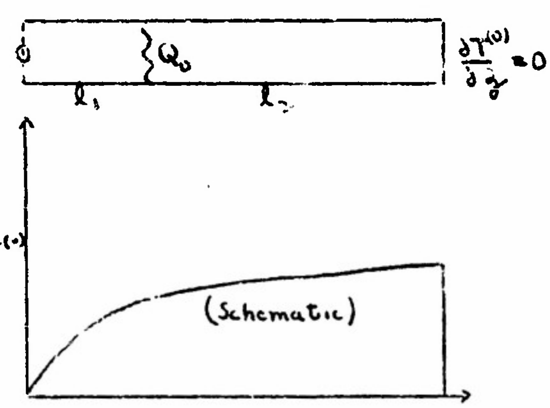
$$T^{(1)} = \frac{T_1(l-z)}{l} + \frac{T_2 z}{l} + \frac{Q_1}{KA}\frac{l_2 z}{l} \;;\; 0 \le z \le l_1 \;; \qquad T^{(0)} = \frac{Q_0}{KA}\frac{\sin \varkappa l_2}{\sin \varkappa l}\frac{\sin \varkappa z}{\varkappa}, \quad 0 \le z \le l_1$$

$$T^{(1)} = \frac{T_1(l-z)}{l} + \frac{T_2 z}{l} + \frac{Q_1}{KA}\frac{l_1(l-z)}{l}, \; l_1 \le z \le l \;; \qquad T^{(0)} = \frac{Q_0}{KA}\frac{\sin \varkappa l_1}{\sin \varkappa l}\frac{\sin \varkappa(l-z)}{\varkappa}, \quad l_1 \le z \le l \tag{3.7}$$

Note that $T^{(1)}$ in (3.7) is identical with (3.2).

Similarly Case 2 separates into the static problem with solution in (3.3) and a dynamic problem with the solution:

Dynamic Problem (case 2)

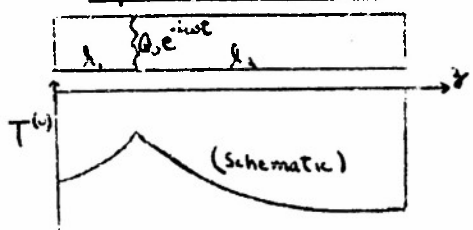


$$T^{(0)} = \frac{Q_0}{KA}\frac{\cos \varkappa l_2}{\cos \varkappa l}\frac{\sin \varkappa z}{\varkappa}, \quad 0 \le z \le l_1$$

$$T^{(0)} = \frac{Q_0}{KA}\frac{\sin \varkappa l_1}{\cos \varkappa l}\frac{\cos \varkappa(l-z)}{\varkappa}, \quad l_1 \le z \le l \tag{3.8}$$

Finally, Case (3) separates into the quasi-static problem (3.4) and the dynamic problem:

Dynamic Problem



(case 3)

$$T^{(\omega)} = -\frac{Q_0}{KA}\frac{\cos\varkappa l_2}{\sin\varkappa l}\frac{\cos\varkappa z}{\varkappa}\,; \quad 0 \le z \le l_1$$

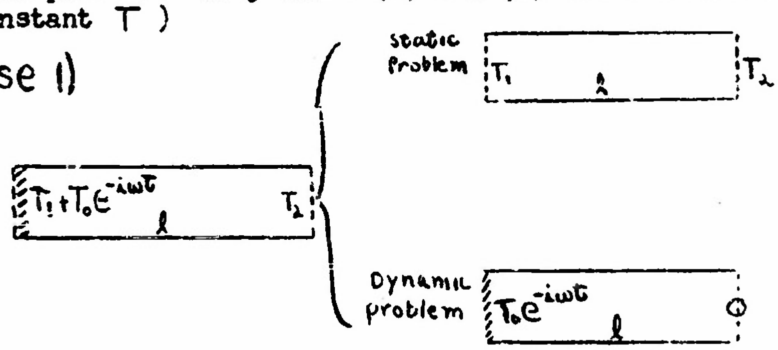
$$T^{(\omega)} = -\frac{Q_0}{KA}\frac{\cos\varkappa l_1}{\sin\varkappa l}\frac{\cos\varkappa (l-z)}{\varkappa}\,, \quad l_1 \le z \le l \tag{3.9}$$

The problems with time varying sources solved above describe the essential physical situation in the heat conduction resulting from resistor sources fed by A.C. Although a simplified source has been used which gives rise only to principal mode waves, less symmetrical sources will behave similarly. They will yield in general both principal and higher mode waves but the latter may be ignored at sufficient distance from the source in comparison with the principal mode because of their heavier attenuation. (See Section 8 for an analysis of the ring source.)

Another and simpler group of non-stationary problems which are not quite as easily realized physically but still may be approached in some circumstances, assumes the temperature of the boundary has a harmonic part as well as a constant part. Again, these problems may be solved simply as the superposition of a static and a dynamic problem. Only Cases (1) and (2) are relevant now (Case (3) has no surface at constant T)

Solutions:

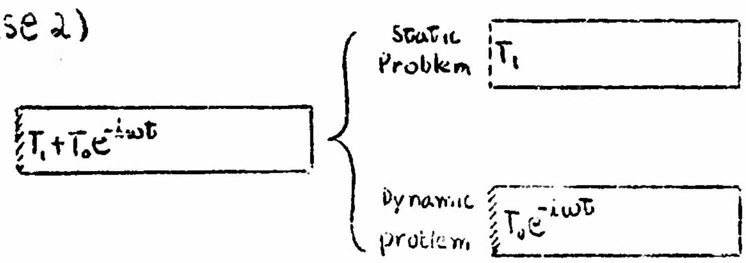
(case 1)



$$T^{(s)} = \frac{T_1 (l-z) + T_2 z}{l}$$

$$T^{(\omega)} = T_0 \frac{\sin\varkappa (l-z)}{\sin\varkappa l} \tag{3.10}$$

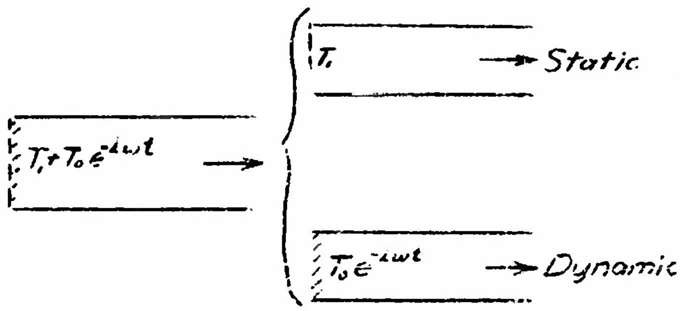
(case 2)



$$T^{(s)} = T_1$$

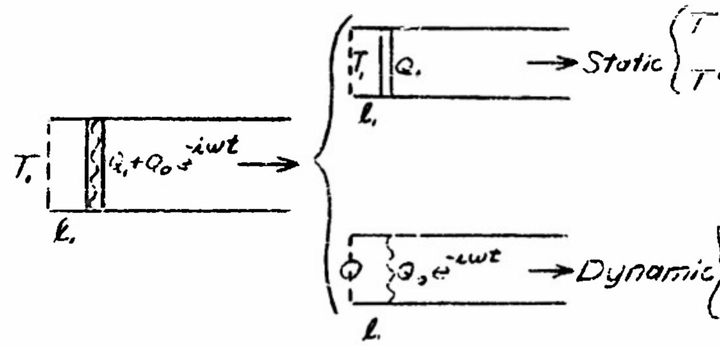
$$T^{(\omega)} = T_0 \frac{\cos\varkappa (l-z)}{\cos\varkappa l} \tag{3.11}$$

Finally, it will be useful to note the solutions to the above source problems with another termination - namely with infinite extension of the conductor in a direction away from the source (indicated by an arrow and open end). There are four such problems:



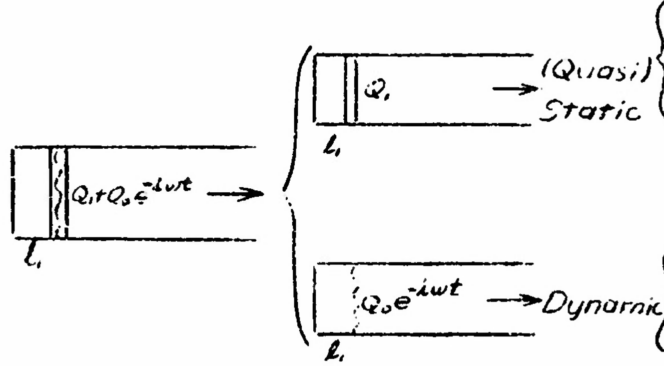
$T^{(1)} \to T_1$ everywhere,
[e.g. $T^{(1)} = T_1\left(1 - \operatorname{erf}\frac{z}{2\sqrt{Dt}}\right)$ if at $t = 0$, $T^{(1)} = 0$ for all z.] (3.12)

$$T^{(0)} = T_0 e^{i\kappa z} \qquad (3.13)$$



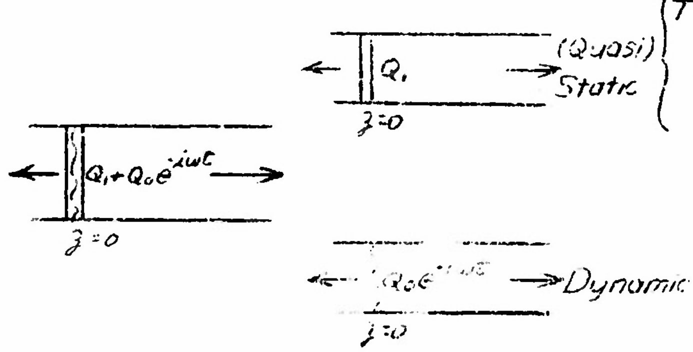
$$\begin{cases} T^{(1)} = T_1 + \dfrac{Q_1}{KA} z, & 0 \le z \le l_1 \\ T^{(1)} \to T_1 + \dfrac{Q_1}{KA} l_1, & l_1 \le z \end{cases} \qquad (3.14)$$

$$\begin{cases} T^{(0)} = \dfrac{Q_0}{KA} e^{i\kappa l_1} \dfrac{\sin \kappa z}{\kappa}, & 0 \le z \le l_1 \\ T^{(0)} = \dfrac{Q_0}{KA} \sin \kappa l_1 \dfrac{e^{i\kappa z}}{\kappa}, & l_1 \le z \end{cases} \qquad (3.15)$$



$$\begin{cases} T^{(1)} = \dfrac{Q_1}{KA}\left[2\left(\dfrac{Dt}{\pi}\right)^{1/2} - \dfrac{l_1}{2}\right], & 0 \le z \le l_1 \\ T^{(1)} \to \dfrac{Q_1}{KA}\left[2\left(\dfrac{Dt}{\pi}\right)^{1/2} - z + \dfrac{l_1}{2}\right], & l_1 \le z \end{cases} \qquad (3.16)$$

$$\begin{cases} T^{(0)} = -\dfrac{Q_0}{KA} e^{i\kappa l_1} \dfrac{\cos \kappa z}{i\kappa}, & 0 \le z \le l_1 \\ T^{(0)} = -\dfrac{Q_0}{KA} \cos \kappa l_1 \dfrac{e^{i\kappa z}}{i\kappa}, & l_1 \le z \end{cases} \qquad (3.17)$$



$$T^{(1)} = \frac{Q_1}{2KA} \int_{|z|}^{\infty} \left(1 - \operatorname{erf}\frac{z}{2\sqrt{Dt}}\right) dz \to \frac{Q_1}{KA}\left[\left(\frac{Dt}{\pi}\right)^{1/2} - \frac{|z|}{2}\right] \text{ at any fixed } z \qquad (3.18)$$

(initially $T^{(1)} = 0$)

$$T^{(0)} = -\frac{Q_0}{KA} \frac{e^{-i\kappa z}}{2i\kappa}, \quad z \le 0$$

$$T^{(0)} = -\frac{Q_0}{KA} \frac{e^{i\kappa z}}{2i\kappa}, \quad z \ge 0 \qquad (3.19)$$

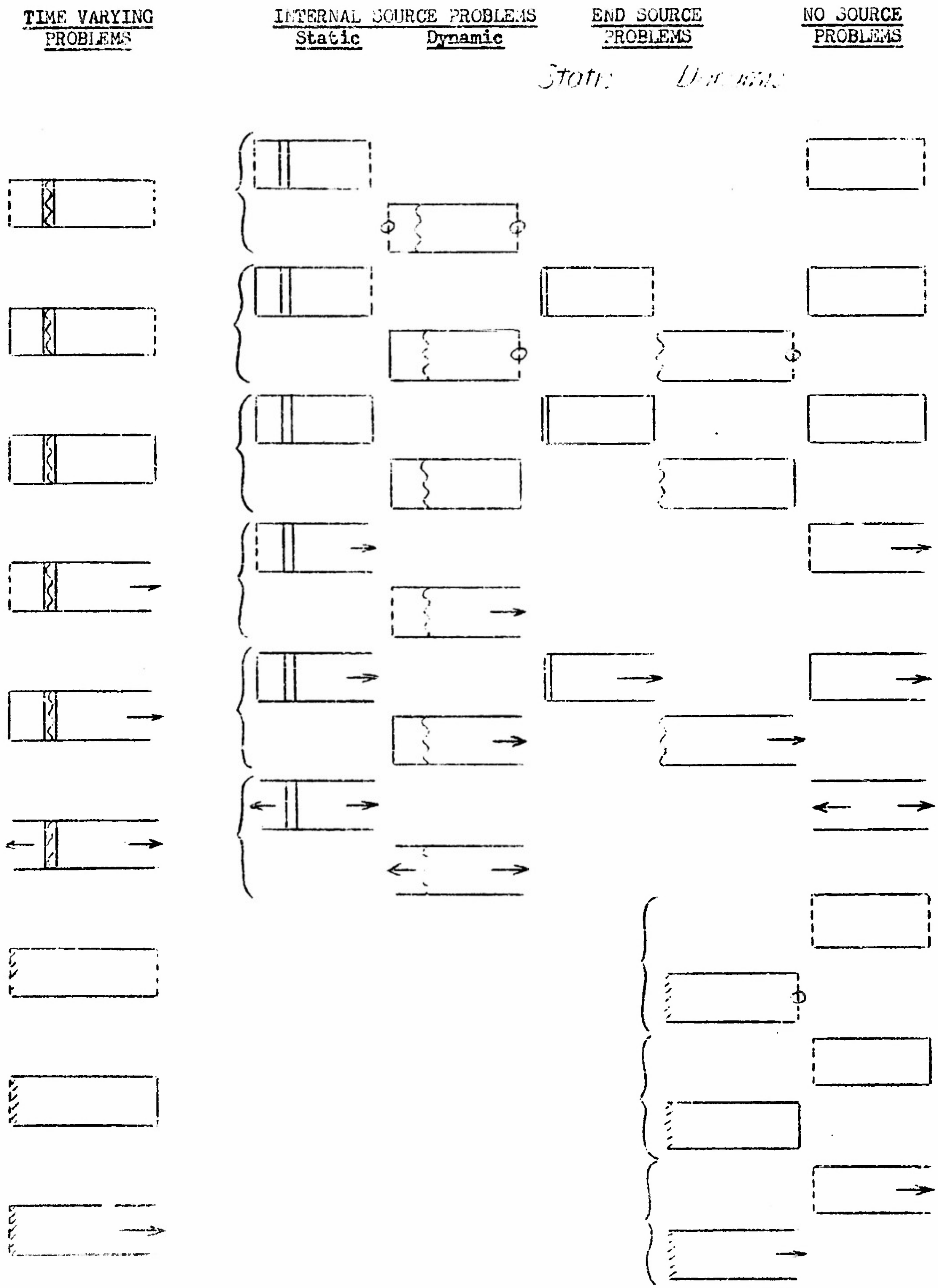
In the static part of all these problems with semi-infinite terminations, the transient dies away in a time depending on the distance from the source or temperature reservoir. Thus the solutions depend on the error function of the argument $(z/2\sqrt{\kappa t})$, where $\operatorname{erf} x = (2/\sqrt{\pi})\int_0^x e^{-y^2}dy$, which rapidly approaches its limiting value of 1 when the argument exceeds 1. Accordingly if at fixed z the temperature approaches a constant (time independent) value for long enough time (the time will be proportional to z^2), we have called the solution static (perhaps asymptotically static would be better). If, however, $T^{(i)}$ approaches a value rising steadily with the same rate at each point, as in (3.16) and (3.18), the solution has been called quasi-static, in analogy with (3.4).

These examples illustrate that various heat flow problems with time varying sources can be solved as the superposition of two problems. One is a dynamic problem with simplified boundary conditions - either T or $(\partial T/\partial n)=0$, in which the periodic solutions are made up of temperature waves driven by the periodic portion of the source. The other is a static problem which keeps the more general boundary condition with respect to T. In certain cases no true static solution exists for this problem, but then a quasi-static state in which the T of all points rises at the same rate does exist. No such difficulties arise in the dynamic problems. In all these cases, the transient solutions which depend on the exact initial conditions, are not of interest here.

(3c) Classification of Problems Treated and Their Equivalent Simpler Problems.

Although all the discussion has been given with simplified sources, uniform over the cross section, this restriction does not affect the essential nature of the solutions under the various conditions considered. The examples discussed above should be typical of all physical situations of heat conduction in cylinders with the boundary surfaces at fixed T, at fixed flux (i.e. fixed value of $\partial T/\partial n$), free $(\partial T/\partial n = 0)$, or with time varying T, or time varying flux. In addition the boundary condition of semi-infinite extension, and the more general case of internal sources have been considered.

The problems treated may be systematically classified as follows, where the solution to the complete time varying problem on the left contains all the simpler problems on the right. Brackets join the two problems (static and dynamic) whose superposition solves the time varying problem. The static problems include internal source problems with all six combinations of the three boundary conditions, fixed T, free, or semi-infinite (||, |:, ::, |→, :→, ↔). In case the internal source occurs with a free boundary, these may be combined to give an end source with the three boundary conditions. When the sources go to zero, the six possible distributions with no sources are obtained. Finally, the time varying boundary T's give on the one hand three static problems which are included in the six no source cases just mentioned, and have fixed T at one end and three boundary conditions at the other, and, on the other hand, three simple dynamic problems, classified under end source problems, since the disturbance is produced at the variable T end. The notation used in the diagrams is the same as that introduced above and is defined again in Appendix 1.


TIME VARYING PROBLEMS
INTERNAL SOURCE PROBLEMS
Static
Dynamic
END SOURCE PROBLEMS
Static
NO SOURCE PROBLEMS

4. THE TRANSMISSION LINE ANALOGY, EQUIVALENT CIRCUITS FOR DISCONTINUITIES AND THE CALCULATION OF REFLECTIONS.

(4a) Transmission Line Equations and the Principal Mode Temperature Wave

The form of the z dependence of any of the temperature modes as either running or standing waves permits immediate representation of these modes by a transmission line. The usefulness of this procedure rests on the practical possibility of considering only the principal mode and its transmission line, since the greater attenuation of the higher modes damps them out relatively quickly. Thus communication between discontinuities such as sources, junctions, or detectors can be considered to take place entirely via the principal mode provided these are spaced sufficiently far apart. The effects of the discontinuities can in fact be represented by a lumped element circuit which simply describes by means of linear relations the net effect of the discontinuity on the lowest mode voltages and currents appearing at the circuit.

We consider now the principal mode and describe its behavior by means of a transmission line - other modes may be treated similarly. In general, (omitting the factor $e^{-i\omega t}$ and dropping superscript $^{(0)}$)

$$T = Ae^{i\mathcal{K}z} + Be^{-i\mathcal{K}z} = \alpha\cos\mathcal{K}z + i\beta\sin\mathcal{K}z$$

$$\frac{\partial T}{\partial z} = i\mathcal{K}(Ae^{i\mathcal{K}z} - Be^{-i\mathcal{K}z}) = \mathcal{K}(i\beta\cos\mathcal{K}z - \alpha\sin\mathcal{K}z) \qquad (4.1)$$

$$\alpha = A + B, \qquad \beta = A - B$$

to be compared with transmission line equations and solutions

$$\frac{\partial V(z)}{\partial z} = i\mathcal{K}Z_0 I(z)\,; \qquad V(z) = V_0\cos\mathcal{K}z + iZ_0 I_0\sin\mathcal{K}z$$

$$\frac{\partial I(z)}{\partial z} = i\mathcal{K}Y_0 V(z)\,; \qquad I(z) = I_0\cos\mathcal{K}z + iY_0 V_0\sin\mathcal{K}z \qquad (4.2)$$

$$V_0 \equiv V(0)\,; \quad I_0 \equiv I(0)\,; \quad Y_0 \equiv (1/Z_0)$$

Two identifications of $V(z)$ (and $I(z)$) are possible, namely with $T(z)$ or with $\partial T/\partial z$ (and, correspondingly, $I(z)$ with $\partial T/\partial z$, or T) There seems to be no important reason to prefer one, although we note that $(\partial T/\partial z)$ has some analogy to the transverse electric field in waveguide, since both vanish at the free surface of a junction with a cylinder of smaller cross section (on the metal wall in the waveguide case, and on the free surface in the present case). Since voltage and electric field

are related, this leads to identification of $(\partial T/\partial z)$ with $V(z)$. However, a different analogy compares $(\partial T/\partial z)$, which is proportional to the heat flux, with the electric current. Thus a free surface is now an open circuit and no heat flux gets through it. Because there is some precedent for the latter choice (private communication from E. Mendoza) we take it as the basis of the analogy. Another advantage is that this will lead to the natural representation of a heat capacity by a capacitative (and resistive) circuit element.

Thus we shall put

$$V(z) = CT = C(\alpha \cos \mathcal{H} z + i\beta \sin \mathcal{H} z) \tag{4.3}$$

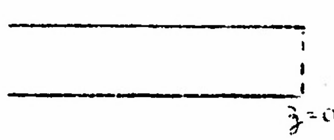
where proportionality constant C is introduced. Comparing (4.3) and (4.2) yields then

$$I(z) = \frac{C}{i\mathcal{H} Z_0} \frac{\partial T}{\partial z} = -iCY_0(i\beta \cos \mathcal{H} z - \alpha \sin \mathcal{H} z) \tag{4.4}$$

where two constants remain to be fixed, C and Y_0. These will be chosen later to simplify the equivalent circuit of the dicontinuity (and to satisfy reciprocity.) $\mathcal{H}$ of course is already fixed and α and β give the wave amplitudes.

(4b) Effects of Discontinuities

To illustrate these equations and to lead up to the description of a junction between different rods, consider the effect of terminating a simple rod in various ways. Thus the termination may be T=0 (the complete problem including the static part, will have T=constant)hence V=0 and this should be considered a short circuit. This gives (taking the origin at the junction)



$$\alpha = 0$$
$$V(z) = Ci\beta \sin \mathcal{H} z$$
$$I(z) = CY_0\beta \cos \mathcal{H} z$$

The reflection coefficient (for the voltage waves) is $R = B/A = (\alpha - \beta)/(\alpha + \beta) = -1$. Similarly, if the termination is a free end, $(\partial T/\partial z) = 0$, hence I=0 and the rod may be considered terminated in an open circuit corresponding to $\beta = 0$. Then the voltage reflection is $R = B/A = +1$. Finally, if the line continues without interruption, i. e. is matched at any point, then B=0, hence $R = 0$, $\alpha = \beta$, $V(z) = C\alpha e^{i\mathcal{H} z}$, $I(z) = CY_0\alpha e^{i\mathcal{H} z}$ and the line may be considered terminated in its characteristic impedance.

In the general case, however, in which two rods or cylinders meet in a junction of arbitrary shape and may in addition have different sizes and materials, (as in Fig. 1) the situation may be represented by a lumped circuit between twotransmission lines with different propagation constants and characteristic impedances,(shown in Fig. 2)

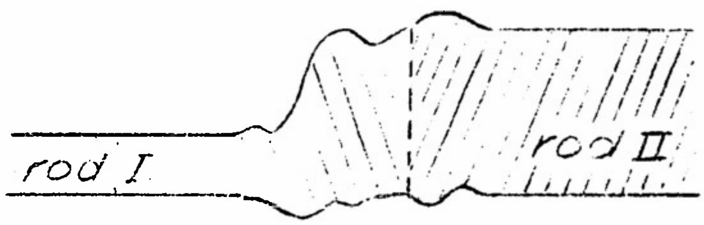


Fig I Change of cross section and (abrupt) change of material at a junction between rods.

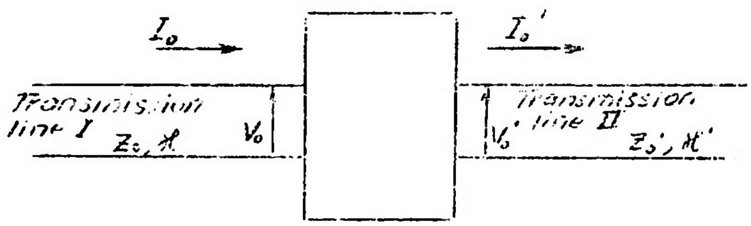


Fig 2 Equivalent transmission lines and lumped circuit for the junction between rods.

To justify the use of the lumped circuit, we note that we can write linear relations among the principal mode amplitudes at certain reference planes in the two rods, in the form

$$\begin{aligned} V_0 &= Z_{11} I_0 + Z_{12} I_0' \\ V_0' &= Z_{21} I_0 + Z_{22} I_0' \end{aligned} \qquad (4.5)$$

(in (4.5) the positive directions of voltage and current in the two lines are taken as in Fig 2. i.e. positive current flow toward increasing z in both lines, positive voltage when the upper conductor has the higher potential. Quantities referring to rod II will be designated by primes.) Such a linear relationship must hold because of the linearity of the entire problem and the fact that two quantities e.g. I_0, I_0' or A, A' completely determine the field distribution. To construct an equivalent

lumped circuit to represent this relation requires in addition that a reciprocity condition be satisfied, which here takes the form

$$Z_{12} = -Z_{21} \qquad (\text{see } 4.14).$$

(4c) <u>Derivation of Reciprocity Condition</u>

The satisfaction of the reciprocity condition is possible in general if the characteristic impedance satisfies a particular condition, which may be found by a procedure like that used in the electromagnetic case. Thus, Green's theorem, applied to any volume V of the conducting rods with surface S gives for arbitrary functions φ_a and φ_b,

$$\int_V (\varphi_a \nabla^2 \varphi_b - \varphi_b \nabla^2 \varphi_a)\, dV = \int_S \left(\varphi_a \frac{\partial \varphi_b}{\partial n} - \varphi_b \frac{\partial \varphi_a}{\partial n}\right) dS \qquad (4.6)$$

Put for φ_a, φ_b in (4.6) two different solutions to the dynamic problem, T_a and T_b, which then satisfy $(\nabla^2 + \mathcal{H}^2)\, T_{a,b} = 0$. Thus the volume integral in (4.6) vanishes, and since $(\partial T_{a,b}/\partial n) = 0$ on the side walls, the only contribution to the surface integral comes from the two cross sections. Let these be at $z = L_1$ and $z = L_2 > L_1$; and substitute from $(4.2), (4.3), (4.4)$,

$$T = \left(\frac{V_o}{c}\right) \cos \mathcal{H} z + (i\, Z_o I_o / c) \sin \mathcal{H} z$$
$$\left(\frac{\partial T}{\partial z}\right) = -\left(\frac{\mathcal{H} V_o}{c}\right) \sin \mathcal{H} z + \left(\frac{i \mathcal{H} Z_o I_o}{c}\right) \cos \mathcal{H} z \qquad (4.7)$$

(4.7) gives the lowest mode forms of T and $(\partial T/\partial z)$ which are dominant sufficiently far from the junction, hence may be assumed to hold at $z = L_1, L_2$. Here V_o and I_o are the values of V and I at the reference plane z=0. Then, substituting (4.7) in (4.6),

$$A\Big[\left(\frac{V_{oa}}{c} \cos \mathcal{H} L_1 + \frac{i Z_o I_{oa}}{c} \sin \mathcal{H} L_1\right)\left(-\frac{\mathcal{H} V_{ob}}{c} \sin \mathcal{H} L_1 + \frac{i \mathcal{H} Z_o I_{ob}}{c} \cos \mathcal{H} L_1\right)$$
$$- \left(\frac{V_{ob}}{c} \cos \mathcal{H} L_1 + \frac{i Z_o I_{ob}}{c} \sin \mathcal{H} L_1\right)\left(-\frac{\mathcal{H} V_{oa}}{c} \sin \mathcal{H} L_1 + \frac{i \mathcal{H} Z_o I_{oa}}{c} \cos \mathcal{H} L_1\right)\Big]$$
$$= \text{similar expression at } z = L_2. \qquad (4.8)$$

where $\left(\frac{\partial T}{\partial n}\right)_{L_1} = -\left(\frac{\partial T}{\partial z}\right)_{L_1}$ has been used, and A is the cross section area of rod I. Now the left side of (4.8) reduces to

$$A \frac{i \mathcal{H} Z_o}{c^2}\left[-I_{oa} V_{ob} + I_{ob} V_{oa}\right] =$$

$$A \frac{i \mathcal{H} Z_o}{c^2} Z_{12}\left[-I_{oa} I_{ob} + I_{ob} I_{oa}\right] \qquad (4.9)$$

where the linear equations(4.5) between V and I at $z=0$ in the two rods have been used . If there is no change of material between the two rods, but just a junction region and possible change of cross-section, then Green's theorem applies at any two cross-sections L_1 ,and L_2 , while (4.9) holds sufficiently far from the junction region. Hence (4.9) would give (note that $T_0, (\partial T_a/\partial z)$ are expressed in terms of V_{oa}, I_{oa} and V'_{oa}, I'_{oa} in the left and right lines respectively)

$$A\frac{i\mathcal{H}Z_0}{c^2}\left[-I_{oa}V_{ob}+I_{ob}V_{oa}\right]=A'\frac{i\mathcal{H}'Z_0'}{(c')^2}\left[-I_{oa}'V_{ob}'+I_{ob}'V_{oa}'\right] \qquad (4.10)$$

or, as in (4.9) on using (4.5)

$$A\frac{i\mathcal{H}Z_0}{c^2}Z_{12}\left[-I_{oa}I_{ob}'+I_{ob}I_{oa}'\right]=A'\frac{i\mathcal{H}'Z_0'}{(c')^2}Z_{21}\left[-I_{oa}'I_{ob}+I_{ob}'I_{oa}\right] \qquad (4.11)$$

Hence

$$A\frac{\mathcal{H}Z_0}{c^2}=A'\frac{\mathcal{H}'Z_0'}{(c')^2}$$ will give the reciprocity

relation $Z_{12} = -Z_{21}$, which is necessary if a lumped circuit representation is to be used.

However, if the material changes, this proof breaks down, since the wave equation $(\nabla^2+\mathcal{H}^2)T=0$ does not hold in the region where the material changes. The proof can be modified if the change is abrupt, taking place at a particular cross-section $z = z_0$, where there may also be an abrupt change of section; then we can write

$$A\frac{i\mathcal{H}Z_0}{c^2}Z_{12}\left[-I_{oa}I_{ob}'+I_{ob}I_{oa}'\right]=\int_{z_0-0}\left(T_a\frac{\partial T_b}{\partial z}-T_b\frac{\partial T_a}{\partial z}\right)dS=$$

$$\frac{K'}{K}\int_{z_0+0}\left(T_a\frac{\partial T_b}{\partial z}-T_b\frac{\partial T_a}{\partial z}\right)dS'=\frac{K'}{K}A'\frac{i\mathcal{H}'Z_0'}{(c')^2}Z_{21}\left[-I_{oa}'I_{ob}+I_{ob}'I_{oa}\right] \qquad (4.12)$$

where use has been made of the continuity of heat flow $K\left(\frac{\partial T}{\partial z}\right)_{z_0-0}=K'\left(\frac{\partial T}{\partial z}\right)_{z_0+0}$ as in (5.5) and that $(\partial T/\partial z)=0$ on the extra (free) walls where one rod ends at the junction.

Thus the general condition which gives reciprocity even for a change of material, provided this happens abruptly at a cross-section, is

$$\frac{K A \mathcal{H} Z_0}{C^2} = \frac{K' A' \mathcal{H}' Z_0'}{(C')^2} \tag{4.13}$$

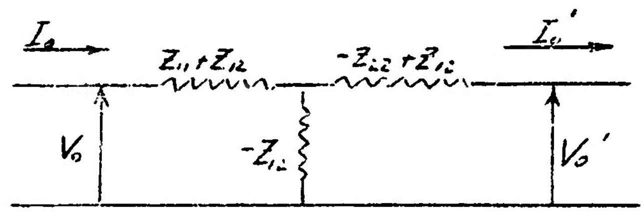
Care will be taken to satisfy this condition later when the change of cross-section in circular guide is explicitly solved. Note, however, that since $\mathcal{H}$ is complex for temperature waves, corresponding to attenuation of the principal mode, the characteristic impedance will also be complex.

(4d) The Equivalent Circuit and the Calculation of Reflection Coefficients

Assuming reciprocity we have

$$V_0 = Z_{11} I_0 + Z_{12} I_0'$$
$$V_0' = -Z_{12} I_0 + Z_{22} I_0' \tag{4.14}$$

with the circuit representation



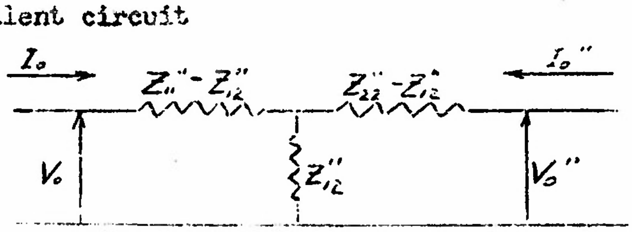
[If the positive direction of current flow in the primed line were to the left, i.e. if we introduce the current

$$I_0'' = \frac{-C'}{i\mathcal{H}' Z_0'}\frac{\partial T}{\partial z} = \frac{C'}{i\mathcal{H}' Z_0'}\frac{\partial T}{\partial z'}$$ where $z' = -z$

so that the z' coordinate has the same relation to the junction as z does, then the sign on the right of (4.11) or (4.12) would be reversed (on using I_0'') and assuming (4.13), we would have reciprocity expressed by $Z_{12} = Z_{21}$.
Hence, circuit equations and equivalent circuit would have the more symmetrical forms

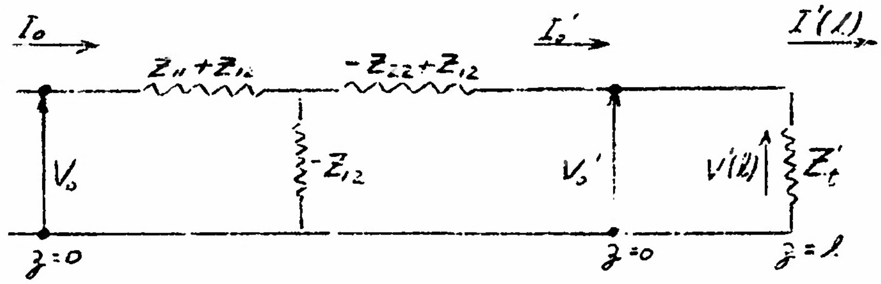
$$V_0 = Z_{11}'' I_0 + Z_{12}'' I_0''$$
$$V_0'' = Z_{12}'' I_0 + Z_{22}'' I_0'' \tag{4.15}$$

with equivalent circuit



where $V_0'' = V_0'$, $I_0'' = -I_0'$, $Z_{11}'' = Z_{11}$, $Z_{12}'' = -Z_{12} = Z_{21} = Z_{21}''$, $Z_{22}'' = -Z_{22}$.]

The equivalent circuit may now be applied to find the reflection of the voltage wave (i.e. the T wave) in the left line due to various terminations, Z_t', of the line on the right. Thus, we have the general circuit



in which the discontinuity at $z=0$ is followed by a length of line l terminated by Z_t'. Then we must have

$$\begin{aligned} V'(l) &= Z_t' I'(l) \\ V_o' &= V'(l)\cos\mathcal{H}'l - i Z_o' I'(l)\sin\mathcal{H}'l \\ I_o' &= I'(l)\cos\mathcal{H}'l - i Y_o' V'(l)\sin\mathcal{H}'l \end{aligned} \qquad (4.16)$$

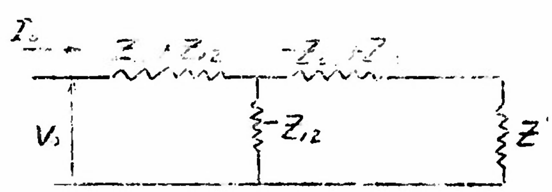
Now, using (4.16) if

$$\frac{V_o'}{I_o'} \equiv Z' = \frac{Z_t' - i Z_o' \tan\mathcal{H}'l}{1 - i Y_o' Z_t' \tan\mathcal{H}'l} \qquad (4.17)$$

then (4.14) gives

$$\frac{V_o}{I_o} = Z_{11} + \frac{Z_{12}^2}{Z_{22} - Z'} \qquad (4.18)$$

(4.18) also follows immediately from the circuit diagram



Now

$$\frac{V_0}{I_0} = \frac{C\alpha}{-iC Y_0 i\beta} = \frac{\alpha}{Y_0 \beta} = \frac{A+B}{Y_0(A-B)} = \frac{1+R}{Y_0(1-R)} \qquad (4.19)$$

where (4.3), (4.4) and (4.1) have been used. (4.17), (4.18) and (4.19) give the general formula for the reflection coefficient

$$\frac{1+R}{1-R} = \frac{Z_{11}}{Z_0} + \frac{(Z_{12}^2/Z_0)}{Z_{22} - \frac{(Z_t' - i Z_0' \tan \varkappa' l)}{(1 - i Y_0' Z_t' \tan \varkappa' l)}} \quad . \qquad (4.20)$$

Certain special cases of (4.20) are of interest, namely

a) Short circuit - $V'(l) = 0$ (ie $T(l) = 0$), hence $Z_t' = 0$, and (4.20) becomes

$$\frac{1+R}{1-R} = \frac{Z_{11}}{Z_0} + \frac{Z_{12}^2}{Z_0(Z_{22} + i Z_0' \tan \varkappa' l)} \qquad (4.21)$$

b) Open circuit - $I'(l) = 0$ (i.e. $(\partial T(l)/\partial z) = 0$), hence $Z_t' = \infty$ and

$$\frac{1+R}{1-R} = \frac{Z_{11}}{Z_0} + \frac{Z_{12}^2}{Z_0(Z_{22} - i Z_0' \cot \varkappa' l)} \qquad (4.22)$$

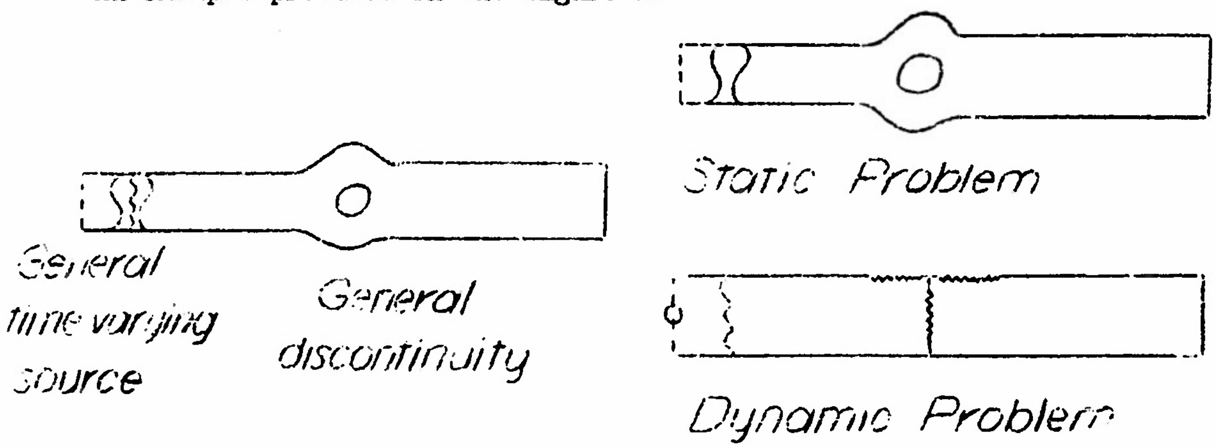
c) Matched termination - (i.e terminated in characteristic impedance or semi-infinite extension of the line), hence $Z_t' = Z_0'$

$$\frac{1+R}{1-R} = \frac{Z_{11}}{Z_0} + \frac{Z_{12}^2}{Z_0(Z_{22} - Z_0)} \qquad (4.23)$$

(4e) General Significance of the Source Problems

Finally, we note that the various heat conduction problems solved in Section 3 which use simplified sources, and uniform cylinders with the three possible terminations (constant T, $\partial T/\partial z = 0$, semi-infinite) may be given a more general significance. Both discontinuities in the cylinder

and general sources may be present, provided communication between these and the boundary cross sections takes place only via the principal mode. An example provided in the figure is



Thus the general problem above separates into a dynamic problem and a static problem. The former consists of the end conditions on the two transmission lines (short circuit left, open circuit right) the lowest mode part of the source, and the equivalent circuit of the discontinuity. The static problem consists of the static boundary conditions ($T = constant$, $\partial T/\partial z = 0$), the steady part of the source, and the general discontinuity. Also, strictly, there remains the higher mode part of the source (time varying) which simply dies away without reaching the discontinuity - hence without generating any more lowest mode contribution. The physical character of all these problems is included in the examples in Section 3, although it may not be possible to write out the solutions explicitly.

(4f) The Capacitative and Resistive Character of Thermal Impedances

The characteristic impedance Z_0 and the constant C in (4.3) are fixed later in (5d) by the requirements that reciprocity be satisfied and that the circuit of the change of cross-section simplifies to a single series element. The result of these requirements is given in (5.15) and makes

$$Z_0 = \left(\frac{1}{KA\varkappa}\right), \qquad C = 1 \tag{4.24}$$

with the natural identification of $I(z)$ with the total heat current down the rod, $Q(z)$, and $V(z)$ with the principal mode part of $T(z)$, $T_p(z)$ (c.f. 5.16). We shall anticipate this identification now, in order to set down in definite form some results of general character on the behavior of the impedance seen by a temperature wave or, briefly, of a thermal impedance

From (4.24), putting in the complex value of $\varkappa$ from (2.13), namely

$$\varkappa = (1+i)\sqrt{\frac{\omega}{2D}} \equiv (1+i)k \tag{4.25}$$

where (4.25) defines the real wave number k, we have

$$Z_o = \frac{(1+i)}{2KAk} . \qquad (4.26)$$

(4.26) shows that Z_o for any rod, of arbitrary thermal conductivity, cross section area, or wave number, is in the first quadrant of the complex impedance plane, on the 45^o line, i. e. that is resistive and capacitative. (note that the choice of the time factor $e^{-i\omega t}$ makes capacities positive imaginary, the opposite of the usual convention for electrical engineering, but more convenient for problems of wave propagation.) This result is readily understandable physically since it means that the current or heat flux builds up before the voltage or temperature (remember the negative time exponent) just as in a condenser the current leads the voltage . Both the heat capacity and the condenser are reservoirs to be filled with heat energy or charge respectively before the corresponding "potential" appears.

A more general result follows by considering the class of impedances produced by transformation of some Z_o' down a length of thermal transmission line, which may have different Z_o and $\mathcal{K}$. From (4.16), this gives rise to impedance Z,

$$Z = \frac{Z_o' - i Z_o \tan \mathcal{K} l}{1 - i Y_o Z_o' \tan \mathcal{K} l} = Z_o \frac{\frac{Z_o'}{Z_o} - \tanh i\mathcal{K} l}{1 - \frac{Z_o'}{Z_o}\tanh i\mathcal{K} l}$$

$$= Z_o \tanh\left(\tanh^{-1}\left(\frac{Z_o'}{Z_o}\right) - i\mathcal{K} l\right) \qquad (4.27)$$

In (4.27) (Z_o'/Z_o) is real and positive. For the case $0 \le Z_o'/Z_o \le 1$, $\tanh^{-1}(Z_o'/Z_o)$ is also real and positive. Since $-i\mathcal{K} l = (1-i)k\,l$, the argument of the last $\tanh$ in (4.27) lies in the 4th quadrant with phase between $-\pi/4$ and 0 . The $\tanh$ of this argument then has the phase limits $-\pi/4$ to .028 radians (1.6^o). This can be shown, for example, by the general relation

$$\tanh(x - iy) = \frac{\sinh 2x - i \sin 2y}{\cosh 2x + \cos 2y} . \qquad (4.28)$$

Multiplying by Z_o with phase $(\pi/4)$, then gives Z the phase 0 to $.813$ radians (46.6^o) (The upper value of this phase comes when $(\sin 2kl/\sinh 2kl)$ which gives the phase of $\tanh(-i\mathcal{K} l)$ has its minimum value at $2kl = 3.93$, just less than $\frac{5\pi}{4}$).

For the case $(Z_o'/Z_o) > 1$

$$\tanh^{-1}(Z_o'/Z_o) = x \pm i\,\pi/2$$

$$\tanh x = Z_o/Z_o' ; \quad 0 \le x \le \infty$$

(the sign of the imaginary term depends on the branch, but does not matter here). Then (4.27) becomes

$$Z = Z_o \tanh\left[x + kl - i(kl \mp \tfrac{\pi}{2})\right] = Z_o \frac{\sinh(2x+2kl) + i \sin 2kl}{\cosh(2x+2kl) - \cos 2kl} . \qquad (4.29)$$

When $\gamma = 0$, corresponding to Z'/Z_0 infinite, and kl is small, (4.29) has the form $Z = Z_0(1+i)/kl$, while for large kl, $Z = Z_0$. Thus Z ranges in phase from $(\pi/2)$ to .758 radians (43.4°).

Hence the transformation of any Z_0' with phase $(\pi/4)$, down any length of line, results always in an impedance remaining in the first quadrant and, in fact, rapidly approaching the value Z_0 of the transforming line. This is shown explicitly by the diagram, where impedance spirals starting from various values in the first quadrant are shown rapidly converging on Z_0, and, of course, always remaining in the first quadrant.

This result suggests the general theorem that all thermal impedances seen by a thermal wave propagating in a rod of arbitrary cross section and material with an arbitrary termination, lie in the first quadrant, hence are resistive and capacitative. Now any impedance in the first quadrant may be realized by proper choice of Z_0', Z_0 and kl, as shown by the discussion above of (4.27) and, particularly, by the diagrams of the impedance spirals produced by transformation down a line. Hence, with this theorem, we have a complete existence theory for thermal impedances.

Such a result is made plausible by the argument that any termination may be considered approximately as a series of thin slices of uniform cross section, hence as a series of sections of thermal transmission lines. Evidently, if we start from some boundary condition such as $T = 0$, or $\partial T/\partial n = 0$, corresponding to impedance 0 or ∞, the result of transforming through the series of transmission line sections remains in the first quadrant by the above discussion. Even a radiation boundary condition (into a vacuum) in which $-K(\partial T/\partial n) = HT$ where H is a positive constant (H. Carslaw and J. Jaeger - Conduction of Heat in Solids, Oxford 1947 p.13), may be included, since this corresponds to an impedance of (1/H) (per unit area) which also starts in the first quadrant.

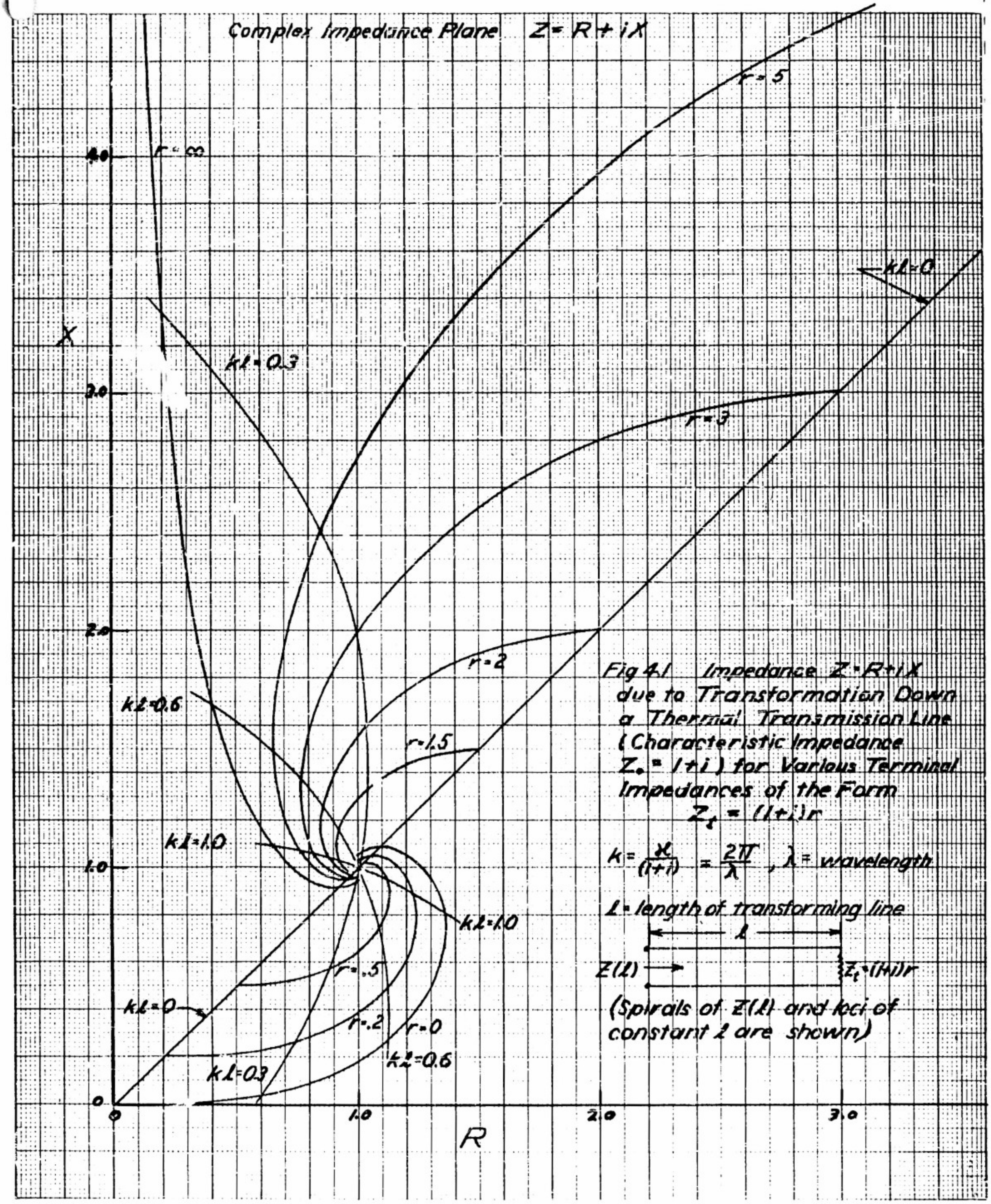


Fig 4.1 Impedance $Z = R + iX$ due to Transformation Down a Thermal Transmission Line (Characteristic Impedance $Z_0 = 1+i$) for Various Terminal Impedances of the Form $Z_t = (1+i)r$

$k = \frac{\varkappa}{(1+i)} = \frac{2\pi}{\lambda}$, λ = wavelength

l = length of transforming line

(Spirals of $Z(l)$ and loci of constant l are shown)

This argument is difficult to make rigorous by passing to the limit of zero thickness slices because it requires consideration of the discontinuity impedances introduced by the successive changes of cross section going from slice to slice. These impedances, which are, in fact, just lumped series impedances, as shown in section 5, do go to zero, as the slices get thinner. To neglect them, however, they really must vanish to higher order than the thickness of the slices. This does appear reasonable from the equations of section 5 for the change of circular cross-section, where (5.33) for Z and (6.6) for ζ_{oo} show that as Γ, the ratio of radii, approaches 1, the factors $J_1(\beta_n/\Gamma)$ appearing in each term of ζ_{oo} approach zero as $(\Gamma-1)^2$, since $J_1(\beta_n)=0$. However ζ_{oo} is only the first term of the variational expression for Z, and consideration of higher terms would be more difficult. In addition, even if this can be carried through, actual abrupt changes of cross section could still be present in the termination and the theorem would have to be proved separately for them. Accordingly we look for another type of proof.

(4g) General Proof that all Thermal Impedances lie in the first Quadrant.

First, we note that the restriction of the impedance implies a restriction on the reflection coefficient $\mathcal{R}$ of a wave incident on that impedance. If Z_t is the terminating impedance and Z_o the characteristic impedance of the line, (4.19) gives

$$\frac{Z_t}{Z_o} = \frac{1+\mathcal{R}}{1-\mathcal{R}} \; ; \qquad \mathcal{R} = \frac{Z_t - Z_o}{Z_t + Z_o} \tag{4.30}$$

The transformation (4.30) takes the complex impedance plane conformally into the complex $\mathcal{R}$ plane, with the first Z_t quadrant going into a portion of the unit circle. Thus putting $Z_o = 1+i$, which loses no generality but just fixes the scale of impedances.

(4.30) becomes

$$Z_t = (1+i)\frac{(1+|\mathcal{R}|e^{i\theta})}{(1-|\mathcal{R}|e^{i\theta})} = \frac{[1-|\mathcal{R}|^2+i(\mathcal{R}-\mathcal{R}^*)]+i[1-|\mathcal{R}|^2-i(\mathcal{R}-\mathcal{R}^*)]}{1+|\mathcal{R}|^2-\mathcal{R}-\mathcal{R}^*} \tag{4.31}$$

where the conjugate of $\mathcal{R}$ is denoted by $\mathcal{R}^* = |\mathcal{R}|e^{-i\theta}$. In (4.31) the denominator

$$1+|\mathcal{R}|^2-\mathcal{R}-\mathcal{R}^* = 1+|\mathcal{R}|^2-2|\mathcal{R}|\cos\theta = (1-|\mathcal{R}|\cos\theta)^2+|\mathcal{R}|^2\sin^2\theta \geq 0 \tag{4.32}$$

hence is always positive. Then if Z_t lies on the positive real axis, we must have

$$1-|R|^2-i(R-R^*) = 1-|R|^2+2|R|\sin\theta = 0 \qquad (4.33a)$$

$$1-|R|^2+i(R-R^*) = 1-|R|^2-2|R|\sin\theta \geq 0 \qquad (4.33b)$$

Putting (4.33a) in (4.33b) gives $-4|R|\sin\theta \geq 0$, hence $\sin\theta \leq 0$ and, solving (4.33a) for $|R|$,

$$|R| = \sqrt{1+\sin^2\theta}+\sin\theta \; ; \quad -\pi < \theta \leq 0 \qquad (4.34)$$

(4.34) gives the locus of R when Z_t is real and positive, and evidently lies on an arc within the unit circle stretching from $+1$ to -1 below the real axis.

Similarly, when Z_t is positive imaginary, the conditions (4.33a) and b are interchanged, and

$$|R| = \sqrt{1+\sin^2\theta}-\sin\theta \; ; \quad 0 \leq \theta \leq \pi \qquad (4.35)$$

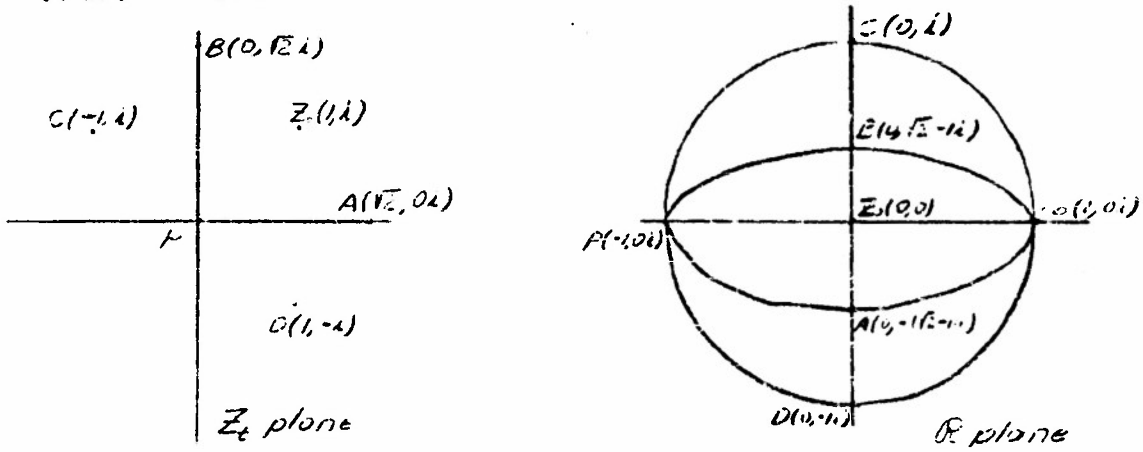
which is an arc obtained by reflecting (4.34) in the real axis.

Evidently the condition that Z_t lies in the first quadrant, requires inequalities for both (4.33) a and b, and leads to

$$|R| \begin{cases} \leq \sqrt{1+\sin^2\theta}-\sin\theta & 0 \leq \theta \leq \pi \\ \leq \sqrt{1+\sin^2\theta}+\sin\theta & -\pi < \theta \leq 0 \end{cases} \qquad (4.36)$$

so that R lies in the elliptical region in the unit circle between the arcs

(4.35) and (4.36). The transformation is illustrated in the diagram



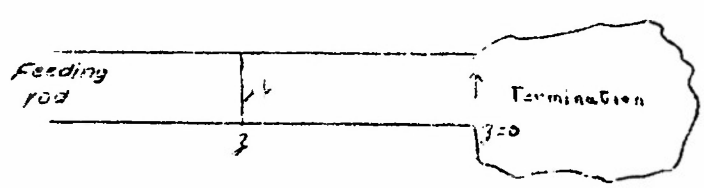
Thus the theorem will follow if we can show generally that the temperature field in the rod has a principal mode of the form $e^{i\kappa z} + R e^{-i\kappa z}$, (taking the termination at $z=0$) where R is restricted in the above fashion.

In the case of electromagnetic wavesin waveguide or acoustical waves in pipes a restriction on R is obtained immediately from conservation of energy. Thus since the reflected wave cannot carry away more energy than the incident wave carries into the termination, and since the energy flow is proportional to the absolute square of the amplitude of a propagating wave, this gives immediately $|R|^2 \le 1$, and R lies within the unit circle, or, correspondingly, Z_r lies in the right half plane when Z_0 is real. For thermal waves, this argument fails, since thermal waves do not carry any energy (the time average heat flux is zero) and would in any case be inadequate since R is restricted more than by $|R| \le 1$. All that one can say by direct physical argument is that the thermal wave represents propagation of a disturbance, and we expect that a disturbance, which here propagates by a diffusion process, must keep diminishing as it spreads and will be reflected in general, with diminished amplitude.

However, the formal proof of the energy theorem from the wave equation and boundary conditions is obtained by applying Green's theorem (4.6) to the tangential electric field and its conjugate. We try a similar procedure here applied to T and T^*. Noting that $\nabla^2 T = -\kappa^2 T$, $\nabla^2 T^* = -\kappa^{*2} T^*$, (4.6) gives

$$(\kappa^2 - \kappa^{*2}) \int_V |T|^2 dV = -\int_A \left(T \frac{\partial T^*}{\partial z} - T^* \frac{\partial T}{\partial z} \right) dA \qquad (4.37)$$

where the volume V includes a section of a rod leading into an arbitrary termination. On all boundaries, either $T=0$ or $(\partial T/\partial n)=0$ except on the cross section A of the feeding rod, hence only that surface integral remains in (4.37), and there $\partial/\partial n = -\partial/\partial z$.



We now put

$$T = e^{i\kappa z} + R e^{-i\kappa z} + \sum_{n=1}^{\infty} A_n e^{-i\kappa_n z} J_0(\gamma_n r) \qquad (4.38)$$

as a general, exact expression for T within the rod in which only a principalmode is incident giving rise to reflected principal mode and higher mode waves. The reflection coefficient is R at cross section $z=0$, which could be at any position along the rod. Then the

termination is defined as the material for $z > 0$, and R is the reflection due to that termination. Although for convenience we have used circular cross section higher modes (with cylindrical symmetry) in (4.36) this is not an essential restriction since the transverse parts of the solution of the wave equation in any cross section form a complete orthogonal set of functions in that cross section, with discrete real characteristic values δ_n which appear in the z dependence through $\mathcal{H}_n^2 = \mathcal{H}^2 - \delta_n^2$.

Returning to (4.37), we put $\mathcal{H} = (1+i)k$ and breakup the integral over V into an integral over the termination and one between z and O, to obtain

$$4k^2 i\Big[\iint_{\text{Termination}} |T|^2 dV + \int_z^0 dz \int_A dA\,|T|^2\Big] = -\int_A dA\, 2i\,\mathcal{I}m\Big(T\frac{\partial T^*}{\partial z}\Big) \qquad (4.39)$$

Now from (4.38)

$$\int_A dA\,|T|^2 = A\Big(e^{-2kz} + |R|^2 e^{2kz} + R e^{-2ikz} + R^* e^{2ikz} + A\sum_{n=1}^{\infty} |A_n|^2 e^{-i(\mathcal{H}_n - \mathcal{H}_n^*)z} J_0^2(\delta_n R)\Big) \qquad (4.40)$$

where use has been made of the orthogonality properties (5.7) of the mode functions in the cross section, and of $A = \pi R^2$.

Also from (4.38)

$$\int_A T\frac{\partial T^*}{\partial z} dA = -A(1+i)k\big(e^{-2kz} - |R|^2 e^{2kz} + R e^{-2ikz} - R^* e^{2ikz}\big) + A\sum_{n=1}^{\infty} i\,\mathcal{H}_n^* |A_n|^2 e^{-i(\mathcal{H}_n - \mathcal{H}_n^*)z} J_0^2(\delta_n R) \qquad (4.41)$$

Hence (4.39) becomes, after integrating (4.40) from z to 0 and noting that the principal mode terms cancel out,

$$1 - |R|^2 - i(R - R^*) = \frac{2k}{A}\int_{\text{Termination}} |T|^2 dV + 2k\sum_{n=1}^{\infty} \frac{|A_n|^2 J_0^2(\delta_n R)}{-i(\mathcal{H}_n - \mathcal{H}_n^*)} \qquad (4.42)$$

(4.42) is independent of z and also follows by putting $z = 0$ in (4.39), (4.41). For $\mathcal{H}_n = [2ik^2 - \delta_n^2]^{1/2}$ where the root with positive imaginary part is chosen (so that $e^{i\mathcal{H}_n z - i\omega t}$ attenuates in the direction of propagation $+z$), hence $\mathcal{H}_n$ must also have a positive real part (which is small for large n).

Then

$$(\mathcal{H}_n + \mathcal{H}_n^*) > 0 \quad ; \quad -i(\mathcal{H}_n - \mathcal{H}_n^*) > 0 \qquad (4.43)$$

$$(\mathcal{H}_n + \mathcal{H}_n^*)(-i[\mathcal{H}_n - \mathcal{H}_n^*]) = 4k^2$$

and (4.42) leads to the conclusion

$$1-|R|^2-i(R-R^*) \equiv 1-|R|^2+2|R|\sin\theta \geq 0 \qquad (4.44)$$

Comparing (4.44) with (4.31)shows that we have proved that Z_t always has positive imaginary part, or lies in the right half plane. For $-\pi \leq \theta \leq 0$, this also gives $Re\, Z_t > 0$ since $Re\, Z_t = 1-|R|^2-2|R|\sin\theta \geq -4|R|\sin\theta$ on using (4.44)
However for $0<\theta<\pi$, $Re\, Z_t$ could be negative; for example if $\theta=\pi/2$, $|R|=\sqrt{2}+1$, then (4.44) is satisfied, and $Re\, Z_t = -4(1+\sqrt{2})$.
Thus only part of the theorem restricting Z_t to the first quadrant has been proved.

The other condition on Z_t is obtained similarly by applying Green's theoremto $(\partial T/\partial z)$ and $(\partial T^*/\partial z)$. In place of (4.37) we obtain

$$(k^2-k^{*2})\int_V |\tfrac{\partial T}{\partial z}|^2 dV = +\int_S \left(\frac{\partial T}{\partial z}\frac{\partial}{\partial n}\left(\frac{\partial T^*}{\partial z}\right) - \frac{\partial T^*}{\partial z}\frac{\partial}{\partial n}\left(\frac{\partial T}{\partial z}\right)\right)dS = -\int_A dA\left(\frac{\partial T}{\partial z}\frac{\partial^2 T^*}{\partial z^2} - \frac{\partial T^*}{\partial z}\frac{\partial^2 T}{\partial z^2}\right) \qquad (4.45)$$

where the reduction of S to just the cross-section A depends on the vanishing of $\partial/\partial n(\partial T/\partial z)$ when either $T=0$ or $(\partial T/\partial n)=0$.

Thus when $T=0$, $\partial T/\partial tan = \partial^2 T/\partial tan^2 = 0$ (derivatives in all tangential directions), hence $(\partial^2 T/\partial n^2) = -k^2 T - k^2(\partial^2 T/\partial tan^2) = 0$.
Therefore $\partial/\partial n(\partial T/\partial z) = \cos(n,z)(\partial^2 T/\partial n^2) = 0$,

where $\cos(n,z)$ is the cosine of the angle between the z axis and the normal. In the other case $(\partial T/\partial n) = 0$, hence $(\partial^2 T/\partial tan\,\partial n) = 0$ Then

$$\frac{\partial}{\partial n}\left(\frac{\partial T}{\partial z}\right) = \frac{\partial}{\partial n}\left(\frac{\partial T}{\partial tan}\cos(tan,z)\right) = \frac{\partial^2 T}{\partial n\,\partial tan}\cos(tan,z) = 0 \text{ in this case too.}$$

In place of (4.39) we now have

$$4k^2\lambda\left[\int_{Termination}|\tfrac{\partial T}{\partial z}|^2 dV + \int_{-\infty}^{0} dz \int_A dA |\tfrac{\partial T}{\partial z}|^2\right] = -2i\int_A dA\, \mathrm{Im}\left(\frac{\partial T}{\partial z}\frac{\partial^2 T^*}{\partial z^2}\right) \qquad (4.46)$$

and putting (4.38) for T in (4.46) gives

$$1-|R|^2+i(R-R^*) = \frac{1}{Ak}\int_{Termination}|\tfrac{\partial T}{\partial z}|^2 dV + \frac{1}{k}\sum_{n=1}^{\infty}\frac{|\varkappa_n|^2|A_n|^2 J_0^2(\delta_n R)}{-i(\varkappa_n-\varkappa_n^*)} \qquad (4.47)$$

Then (4.43) and (4.47) lead to

$$1-|R|^2+i(R-R^*) \equiv 1-|R|^2-2|R|\sin\theta \geq 0 \tag{4.48}$$

which with (4.31) and (4.44) completes the proof that Z_t always lies in the first quadrant.

The significance of the conditions (4.48) and (4.44) is further brought out by the behavior of the magnitudes of T_p and $(\partial T/\partial z)_p$ as functions of z. T_p and $(\partial T/\partial z)_p$ are the lowest mode parts of T, and are also the mean values of T and $(\partial T/\partial z)$ over the cross section in view of the orthogonality relations of the higher mode functions. $(\partial T/\partial z)_p$ is thus proportional to the net heat flow through a cross section, as shown in (5.14).

It follows easily from $T_p = e^{i\kappa z} + R e^{-i\kappa z}$ that

$$\frac{d|T_p|^2}{dz} = -2k\left[e^{-2kz} - |R|^2 e^{2kz} + i\left(R e^{-2ikz} - R^* e^{2ikz}\right)\right] \tag{4.49}$$

$$\frac{d}{dz}\left|\frac{\partial T_p}{\partial z}\right|^2 = -4k^3\left[e^{-2kz} - |R|^2 e^{2kz} - i\left(R e^{-2ikz} - R^* e^{2ikz}\right)\right] \tag{4.50}$$

Thus at $z=0$, (4.48) and (4.44) reduce to the conditions that

$$\frac{d|T_p|^2}{dz} \leq 0, \qquad \frac{d}{dz}\left|\frac{\partial T_p}{\partial z}\right|^2 \leq 0. \tag{4.51}$$

Since $z=0$ was an arbitrary position in the rod, this means that the mean values of T_p and $(\partial T/\partial z)_p$ continually decrease in magnitude as we move away from the source, and nothing like a node or minimum in the magnitude of the field can occur. Thus a resonant condition and true standing waves can never occur for temperature waves, as we might also expect from the fact that only capacitative, but no inductive reactances are available for these waves. Hence no resonant combination with zero reactance can be put together.

A direct proof that (4.51) holds may also be developed by relating the values of T_p and $(\partial T_p/\partial z)$ in the rod to those at the boundary of the termination. One considers the mean values of T and $(\partial T/\partial z)$ over cross sections of an extension of the rod into the material of the termination. A theorem is easily proved that shows these mean values can never have a maximum in the conducting material. Hence if the values increase going into the material at the boundary, they must increase all the way to the source, which gives (4.51). At the boundary if we start, for example, with the condition $T=0$, then $|T|$ must increase on going into the material.

The theorem about the maximum follows simply from the identity

$$\bar{T}(z) = \frac{1}{\cosh i\kappa l}\left(\frac{\bar{T}(z+l)+\bar{T}(z-l)}{2}\right) \qquad (4.52)$$

where $\bar{T}(z)$ is the mean value of $T(z)$ over the cross section of the rod or the extension into the termination. (4.52) comes from Green's theorem applied to $T(z)$ and $e^{\pm i\kappa(z-\bar{z})}$, (the cross sections must lie completely in the material.)

Various complications develop when the cross section intersects the boundary, and in the consideration of $(\partial T/\partial z)$ with boundary condition $T=0$ and vice versa, that make this argument difficult to carry through, and the method given above seems preferable.

5. THE CHANGE OF CROSS SECTION FOR CIRCULAR CYLINDERS

(5a) General Outline and Background

We now solve explicitly the problem of the temperature distribution around a junction between circular and cylindrical rods of different diameters and materials. As indicated previously this will separate into a static and a dynamic problem. We are primarily interested in the dynamic problem, namely reflection and transmission of a principal mode temperature wave at the junction, since it is this wave on which measurements are made in the experimental technique for determining diffusivities.

In the following we separate the static and dynamic problems. The latter is solved by first expressing the temperature fields in characteristic mode expansions, then matching the expansions at the junction plane z=0 and obtaining an integral equation for $(\partial T/\partial z)$ in the cross section of that plane. Voltages, currents and characteristic impedances for the principal mode behaviors in the two rods are defined in accord with the transmission line analogy of the last section. The integral equation leads directly to a variational expression for a series impedance, Z, representing the effect of the junction plane on the principal mode amplitudes, i.e. voltages and currents, in the rods. Expanding the function $(\partial T/\partial z)$ in modes of the first rod, substituting the expansion into the expression for Z and usung its variational character, gives an infinite set of linear equations for the mode amplitudes. In turn Z is expressed in terms of these mode amplitudes. The numerical evaluation of Z, for appropriate constants, is then carried out in the next section.

Analogous electromagnetic problems are concerned with propagation of radiation in a wave guide and through a wave-guide junction. The formulation of these problems, which introduces a transmission line analogy and obtains variational expressions for the elements of the equivalent circuit representing the junctions, is due to J. Schwinger.[2][3] Later work along these lines is also relevant.[4][5][6]

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(2) J. Schwinger - Unpublished lectures at the M. I. T. Radiation Laboratory, 1943-46.

(3) D. Saxon - "Notes on Lectures by J. Schwinger, Discontinuities in Waveguides", Feb. 1945 (develops transmission line and equivalent circuit analogy, solves a number of simple discontinuity and junction problems by variational formulation and the static method, including capacitive diaphragms and the change in height of a rectangular guide.)

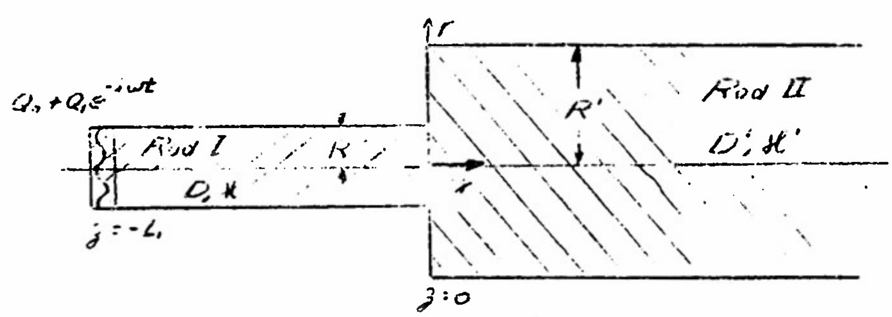
(4) "The Waveguide Handbook," edited by N. Marcuvitz, McGraw-Hill Book Co. (1949) (tabulates equivalent circuit elements for a great variety of waveguide discontinuities and junctions, discusses the transmission line analogy, but does not give any mathematical methods.)

(5) L. Lewin - "Advanced Theory of Waveguides," Iliffe and Sons, Ltd. London (1951) (Presents some of the mathematical methods of solving some of the discontinuity problems, including E plane and H plane steps; gives extensive bibliography)

(6) J. Miles - J. Acoust. Soc. Am. 16, 14, (1944), 17, 279 (1946) (sets up the problem of reflection of principal mode sound waves in a circular tube at a change of cross section, giving variational formula for equivalent circuit element; no numerical results or calculation methods given)

The present problem differs from the electromagnetic waveguide problem (and the acoustical problem of sound propagation in pipes) in the following ways. First, the quantity sought is a scalar field (as in the acoustical case) rather than a vector field (in fact the electromagnetic problem seeks two vector fields E and H) so that the differential equations, characteristic modes and boundary conditions are all much simpler. (Ultimately, however, only problems reducing to one dimensional problems in a junction plane or on a surface of discontinuity can be solved, in the electromagnetic case as well.) Second, the principal mode field is constant over the cross section and has no cutoff (i.e. minimum) frequency, a situation analogous to the acoustical problem, but unlike the electromagnetic case. Third, all modes, including the principal mode, have complex propagation constants, so that even the principal mode attenuates and the corresponding transmission line has a complex characteristic impedance. This situation is unlike both the electromagnetic and acoustical problems. Fourth, the actual physical problem here separates into the superposition of a static and a dynamic problem, with solutions of quite different character, of which only the latter is represented by the transmission line analogy. Finally, we note that the problem of the change of cross section, as solved here by the use of a variational expression for the equivalent series impedance, includes the possibility of an abrupt change of material at the junction plane, a generalization of more importance for this problem than for the electromagnetic or acoustical problem.

(5b) Description of Physical Situation and Form of Solution



As in the diagram, rod I with radius R and constants $\mathcal{K}$ and D is at the left; rod II is at the right with the corresponding quantities R′ , $\mathcal{K}'$, D' designated by primes as in Section 3; rod II will by convention always have the larger radius. The junction plane is at $z = 0$, where the changes of cross-section and material take place.

To illustrate a definite physical situation we assume a time varying source of heat at a free end at $z = -L_1$ in rod I, sufficiently far from the junction so that only the principal mode communicates between source and junction, and let rod II extend to infinity. Then, as shown in Section 2, the solution to this problem is the superposition of two problems. One is a static problem (actually quasi-static) with a steady heat source Q_0 at $z = -L_1$ and a steady heat current flowing into and past the junction, while T satisfies Laplace's equation. The other is a dynamic problem with source $Q_1 e^{-i\omega t}$ at $z = -L_1$, and T satisfies the wave equation $(\nabla^2 + \mathcal{K}^2) T = 0$.

Restricting our attention for now only to the dynamic problem, we expect T in rod I to consist of travelling principal mode waves in both directions, and in rod II of outgoing principal mode waves only. The situation in rod I is described completely by the reflection coefficient $\mathcal{R}$ seen by a wave approaching the junction (when rod II has a semi-infinite or matched termination). $\mathcal{R}$ can be calculated from a knowledge of the equivalent circuit of the junction, as shown in Section 4. The fields in rod I then take the forms

$$V(z) = CT(z) = -\frac{CQ_1 i}{KA\varkappa}\,\frac{e^{i\varkappa z} + Re^{-i\varkappa z}}{Re^{i\varkappa L_1} - e^{-i\varkappa L_1}}$$

$$I(z) = \frac{C}{i\varkappa Z_0}\frac{\partial T}{\partial z} = -\frac{CQ_1 i}{KA\varkappa Z_0}\,\frac{e^{i\varkappa z} - Re^{-i\varkappa z}}{Re^{i\varkappa L_1} - e^{-i\varkappa L_1}} \tag{5.1}$$

which agree with (4.4), have voltage reflection coefficient R at $z = 0$ and satisfy

$$-KA\left(\frac{\partial T}{\partial z}\right)_{z=-L_1} = \frac{-KA\,I(-L_1)\,i\varkappa Z_0}{C} = Q_1\,. \tag{5.2}$$

When $|e^{2i\varkappa L_1}| \ll 1$, (5.1) takes the form

$$V(z) = \frac{CQ_1 e^{i\varkappa L_1}}{KA\varkappa}\left(e^{i\varkappa z} + Re^{-i\varkappa z}\right)$$

$$I(z) = \frac{CQ_1 i}{KA\varkappa Z_0}\,e^{i\varkappa L_1}\left(e^{i\varkappa z} - Re^{-i\varkappa z}\right) \tag{5.3}$$

(5.3) is immediate if the assumption is made that the reflection at the source end of the reflected wave can be neglected.

We proceed now to the rigorous formulation and solution of the problem, which will lead to a circuit representation, hence to the reflection coefficient of the junction.

(5c) The Mode Expansions

Expand $(\partial T/\partial z)$ in rods I and II in the complete set of modes with circular symmetry, as found in (2.12), but taking only principal modes incident from left and right. Then

$$\left(\frac{\partial T}{\partial z}\right)^{I} = \alpha \sin\varkappa z + \sum_{n=0}^{\infty} A_n J_0(\sigma_n r)\, e^{-i\varkappa_n z}$$

$$\left(\frac{\partial T}{\partial z}\right)^{II} = \alpha' \sin\varkappa' z + \sum_{n=0}^{\infty} A'_n J_0(\sigma'_n r)\, e^{i\varkappa'_n z} \tag{5.4}$$

where we have introduced only higher modes which go away from the junction, and $\mathcal{K}, \mathcal{K}', \mathcal{L}_n, \mathcal{H}_n, \mathcal{H}_n', \gamma_n, \gamma_n'$ are defined in (2.9), (2.11). We note that $(\partial T/\partial z)^{II} = 0$ on the annular metal surface between R and R' in the plane z=0. Hence in that plane, values of $(\partial T/\partial z)$ exist only in the circular region (radiu R)common to the two rods. This is why the basic expansion (5.4) is given for $(\partial T/\partial z)$ rather than for T. Now the values of $(\partial T/\partial z)$ at the junction satisfy

$$K\left(\frac{\partial T}{\partial z}\right)^{I}_{z=0} = K'\left(\frac{\partial T}{\partial z}\right)^{II}_{z=0} \tag{5.5}$$

since heat flow across the junction is conserved. Call the function in (5.5) E(r). Then

$$E(r) = K\sum_{n=0}^{\infty} A_n J_0(\gamma_n r) = K'\sum_{n=0}^{\infty} A_n' J_0(\gamma_n' r) \tag{5.6}$$

Now making use of the orthogonality properties of the $J_0(\gamma_n r)$, namely

$$\int_0^R J_0(\gamma_m r)J_0(\gamma_n r)r\,dr = \left[\frac{\gamma_n r J_0(\gamma_m r)J_1(\gamma_n r) - \gamma_m r J_1(\gamma_m r)J_0(\gamma_n r)}{\gamma_m^2 - \gamma_n^2}\right]_0^R = 0 \text{ for } m \neq n$$

$$\int_0^R J_0^2(\gamma_m r)r\,dr = \frac{R^2 J_0^2(\gamma_m R)}{2} \tag{5.7}$$

we can evaluate A_n and A_n' by multiplying (5.6) by $rJ_0(\gamma_n r)$ or $rJ_0(\gamma_n' r)$ and integrating from 0 to R or 0 to R' respectively, giving

$$A_n = \frac{2}{KR^2 J_0^2(\gamma_n R)}\int_0^R E(r)J_0(\gamma_n r)r\,dr\,; \qquad A_n' = \frac{2}{K'R'^2 J_0^2(\gamma_n' R')}\int_0^R E(r)J_0(\gamma_n' r)r\,dr \tag{5.8}$$

Integrating (5.4)

$$T^{I} = -\frac{\alpha}{\mathcal{H}}\cos\mathcal{H}z + \sum_{n=0}^{\infty}\frac{A_n}{-i\mathcal{H}_n}J_0(\gamma_n r)e^{-i\mathcal{H}_n z}$$

$$T^{II} = \frac{-\alpha'}{\mathcal{H}'}\cos\mathcal{H}'z + \sum_{n=0}^{\infty}\frac{A_n'}{+i\mathcal{H}_n'}J_0(\gamma_n' r)e^{i\mathcal{H}_n' z} \tag{5.9}$$

where the constant of integration, independent of z, is discarded because it

does not satisfy the differential equation $((\nabla^2+\varkappa^2)T=0)$.

Substituting the lowest mode terms in (5.9) by $V(z)/C$, and $V'(z)/C'$ gives

$$T^{I}=\frac{V(z)}{C}-\sum_{n=1}^{\infty}\frac{A_n}{i\varkappa_n}J_0(\gamma_n r)e^{-i\varkappa_n z}\;;\qquad \frac{V(z)}{C}=-\frac{\alpha}{\varkappa}\cos\varkappa z-\frac{A_0}{i\varkappa}e^{-i\varkappa z}$$

$$T^{II}=\frac{V'(z)}{C'}+\sum_{n=1}^{\infty}\frac{A_n'}{i\varkappa_n'}J_0(\gamma_n' r)e^{i\varkappa_n' z}\;;\qquad \frac{V'(z)}{C'}=-\frac{\alpha'}{\varkappa'}\cos\varkappa' z+\frac{A_0'}{i\varkappa'}e^{i\varkappa' z} \tag{5.10}$$

and similarly in (5.4) introducing $I(z)$, $I'(z)$, gives

$$\left(\frac{\partial T}{\partial z}\right)^{I}=\frac{i\varkappa Z_0 I(z)}{C}+\sum_{n=1}^{\infty}A_n J_0(\gamma_n r)e^{-i\varkappa_n z}\;,\qquad \frac{i\varkappa Z_0 I(z)}{C}=\alpha\sin\varkappa z+A_0e^{-i\varkappa z}$$

$$\left(\frac{\partial T}{\partial z}\right)^{II}=\frac{i\varkappa' Z_0' I'(z)}{C'}+\sum_{n=1}^{\infty}A_n' J_0(\gamma_n' r)e^{i\varkappa_n' z}\;;\qquad \frac{i\varkappa' Z_0' I'(z)}{C'}=\alpha'\sin\varkappa' z+A_0'e^{i\varkappa' z} \tag{5.11}$$

(5d) Fixing the Characteric Impedance

The constants C, Z_0, C', Z_0' are thus far arbitrary in the formulation of the transmission line analogies for rods I and II. Simple additional requirements will fix their values, however. First, we wish to represent the effect of the discontinuity by a lumped circuit inserted between the two transmission lines. However, a lumped circuit necessarily implies that the linear relations between V_0, V_0', I_0, I_0' satisfy reciprocity, hence as shown in Section (4b), we must have (4.13),

$$\frac{KA\varkappa Z_0}{C^2}=\frac{K'A'\varkappa' Z_0'}{(C')^2}. \tag{5.12}$$

Second, we wish to simplify the equivalent circuit as much as possible. Now we see from (5.11) that by choosing the natural reference plane $z=0$ at which to evaluate currents and voltage, we can make $I_0=I_0'$ for all conditions of excitation (any combination of principal mode waves incident on the junction), since at $z=0$, α and α' drop out of the equations for $I(z)$ and $I'(z)$. Then, taking A_0 and A_0' from (5.8) and noting that the common factor, $\int_0^{R} E(r)\,r\,dr$ cancels out, we obtain from the requirement $I_0=I_0'$

$$\frac{KR^2\varkappa Z_0}{C}=\frac{K'R'^2\varkappa' Z_0'}{C'}. \tag{5.13}$$

Now $A=\pi R^2$, $A'=\pi R'^2$, hence dividing (5.13) by (5.12) gives $C=C'$,

and thus C is a universal constant, independent of the special properties of a particular rod.

(5.12) and (5.13) are then satisfied by choosing Z_0 proportional to $(1/KA\mathcal{H})$. The proportionality constant may now be fixed by the requirement that $I(z)$ be simply the total heat flux, Q, passing through any cross section of the rod. We have

$$Q = -K\int_A \frac{\partial T}{\partial z}\,dS = \frac{-KAi\mathcal{H}Z_0}{C}\,I(z) \qquad (5.14)$$

on using (5.11) and noting that by (5.7) $\int_0^R r J_0(\gamma_n r)\,dr = 0$, so that only the principal mode contributes to the heat flux through a cross section. Hence, if we choose

$$Z_0 = \frac{i}{KA\mathcal{H}}, \qquad C = 1 \qquad (5.15)$$

we have the simple relations from (4.3) and (5.14)

$$\begin{aligned} V(z) &= T_p(z) \\ I(z) &= -KA\left(\frac{\partial T}{\partial z}\right)_p = Q(z) \end{aligned} \qquad (5.16)$$

where $T_p(z)$ and $(\partial T/\partial z)_p$ refer to the principal mode parts of $T(z)$ and $(\partial T/\partial z)$.

Since Q must be continuous across each cross section, including the one at $z = 0$, (5.14) and the requirement that $I(0)$ be continuous lead immediately to (5.13) and, in fact, provide an alternative derivation of (5.13).

The continuity of $I(0)$ means that the equivalent lumped circuit is simply a series impedance, with no elements in shunt across the line, and explicit expressions for the series element will be derived below. Note that it is not possible to make $V(0)$ continuous, since by (5.10) the coefficients α and α' do not drop out at $z = 0$, and these vary with the excitation conditions. However $T(0)$ must be continuous over the region $0 \le r \le R$, so that we must have, by (5.10)

$$\left.\begin{aligned} T^{I}(0) &= T^{II}(0) \\ V(0) - V'(0) &= \sum_{n=1}^{\infty} \frac{A_n}{i\mathcal{H}_n} J_0(\gamma_n r) + \sum_{n=1}^{\infty} \frac{A_n'}{i\mathcal{H}_n'} J_0(\gamma_n' r) \end{aligned}\right\} \; 0 \le r \le R \qquad (5.17)$$

(5e) The Equivalent Series Element and the Variational Expression

Now the discontinuity in $V(0)$, i.e. $V_0 - V_0'$, must be proportional to the single value of $I(0)$ which measures the lowest mode amplitude of E(r), since all equations of the problem are linear. Increasing E(r) by a factor will increase $I(0)$ and $V(0) - V'(0)$ by the same factor. Hence we may set

$$V(0) - V'(0) = Z\, I(0) = -2\pi Z \int_0^R r E(r)\, dr \qquad (5.18)$$

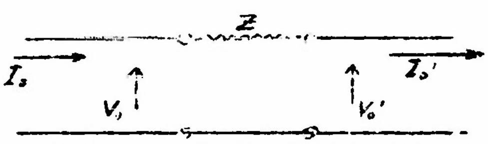
thus defining an impedance Z which is independent of the excitation conditions and determined the overall effect of the obstacle on the lowest mode. Then (5.17) becomes

$$Z = \frac{1}{2\pi \int_0^R E(r)\, r\, dr}\left[\frac{2}{KR^2}\sum_{n=1}^{\infty}\frac{J_0(\gamma_n r)}{-i\kappa_n J_0^2(\gamma_n R)}\int_0^R E(r) J_0(\gamma_n r)\, r\, dr \right.$$

$$\left. + \frac{2}{K'R'^2}\sum_{n=1}^{\infty}\frac{J_0(\gamma_n' r)\int_0^R E(r) J_0(\gamma_n' r)\, r\, dr}{-i\kappa_n' J_0^2(\gamma_n' R')}\right] . \qquad (5.19)$$

(5.19) is a homogeneous integral equation for E(r), for which Z is the eigenvalue.

The relation (5.18) clearly has the following circuit representation



This has a single series impedance of magnitude Z in the line, as expected from the continuity of I_0.

From (5.19) we obtain a variational expression (i.e. stationary for small changes of E(r)) for Z in terms of E(r), by multiplying by $E(r)\, r\, dr$ and integrating over 0 to R,

$$Z = \frac{1}{\pi\left[\int_0^R E(r)\, r\, dr\right]^2}\left[\frac{1}{KR^2}\sum_{n=1}^{\infty}\frac{1}{-i\kappa_n J_0^2(\gamma_n R)}\left(\int_0^R E(r) J_0(\gamma_n r)\, r\, dr\right)^2 + \frac{1}{K'R'^2}\sum_{n=1}^{\infty}\frac{1}{-i\kappa_n' J_0^2(\gamma_n' R')}\left(\int_0^R E(r) J_0(\gamma_n' r)\, r\, dr\right)^2\right]$$

$$(5.20)$$

The stationary character of Z is easily proved by "differentiating" (5.20) with respect to $E(r)$ (i.e. taking the functional variation of E(r)).

Thus we write (5.19) in the schematized form. (the H_n stand for the constant coefficients in (5.19))

$$Z \int_0^R E(r)\, r\, dr = \sum_n H_n J_0(\gamma_n r) \int_0^R E(r) J_0(\gamma_n r)\, r\, dr \qquad (5.21)$$

and (5.20) in the form

$$Z\left(\int_0^R E(r)\, r\, dr\right)^2 = \sum_n H_n \left(\int_0^R E(r) J_0(\gamma_n r)\, r\, dr\right)^2 . \qquad (5.22)$$

Taking the variation δE of E and δZ of Z, (5.22) becomes

$$\delta Z\left(\int E(r)r\,dr\right)^2+2Z\left(\int E\,r\,dr\right)\left(\int\delta E\,r\,dr\right)=2\sum_n H_n\left(\int E J_0 r\,dr\right)\left(\int\delta E J_0\,r\,dr\right)$$

which gives

$$\delta Z\left(\int E r\,dr\right)^2=2\int\delta E r\,dr\left[-Z\int E\,r\,dr+\sum_n H_n J_0\int E J_0 r\,dr\right]=0 \qquad (5.23)$$

where the right hand side of (5.23) vanishes by the original integral equation (5.21)

We now rewrite (5.20) in terms of the coefficients of the higher modes using (5.8) and (5.6). We have

$$\frac{1}{KR^2}\sum_{n=1}^{\infty}\frac{1}{-i\mathcal{H}_n J_0^2(\gamma_n R)}\left(\int_0^R E(r)J_0(\gamma_n r)r\,dr\right)^2=\frac{KR^2}{4}\sum_{n=1}^{\infty}\frac{J_0^2(\gamma_n R)A_n^2}{-i\mathcal{H}_n} \qquad (5.24)$$

$$\frac{1}{K'R'^2}\sum_{n=1}^{\infty}\frac{1}{-i\mathcal{H}_n' J_0^2(\gamma_n' R')}\left(\int_0^R E(r)J_0(\gamma_n' r)r\,dr\right)^2=$$

$$\frac{1}{K'R'^2}\sum_{n=1}^{\infty}\frac{K^2}{-i\mathcal{H}_n' J_0^2(\gamma_n' R')}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}A_\ell A_m\int_0^R J_0(\gamma_\ell r)J_0(\gamma_n' r)r\,dr\int_0^R J_0(\gamma_m r)J_0(\gamma_n' r)r\,dr=$$

$$=\frac{KR^2}{4}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}A_\ell A_m C_{\ell m} \qquad (5.25)$$

where

$$C_{\ell m}=\sum_{n=1}^{\infty}\frac{D_{\ell n}D_{mn}}{-i\mathcal{H}_n'}=C_{m\ell}\ ;\quad D_{\ell n}=\frac{2}{RR'}\left(\frac{K}{K'}\right)^{\frac{1}{2}}\frac{1}{J_0(\gamma_n' R')}\int_0^R J_0(\gamma_\ell r)J_0(\gamma_n' r)r\,dr \qquad (5.26)$$

hence, using (5.7)

$$D_{\ell n} = \frac{2}{R'}\left(\frac{K}{K'}\right)^{\frac{1}{2}} \frac{\gamma_n'}{\gamma_n'^2 - \gamma_\ell^2} \frac{J_0(\gamma_\ell R) J_1(\gamma_n' R)}{J_0(\gamma_n' R')} \tag{5.27}$$

Then

$$D_{0n} = \frac{2}{R'}\left(\frac{K}{K'}\right)^{\frac{1}{2}} \frac{J_1(\gamma_n' R)}{\gamma_n' J_0(\gamma_n' R')} \quad , \quad D_{1n} = \frac{2}{R'}\left(\frac{K}{K'}\right)^{\frac{1}{2}} \frac{\gamma_n'}{\gamma_n'^2 - \gamma_1^2} \frac{J_0(\gamma_1 R) J_1(\gamma_n' R)}{J_0(\gamma_n' R')} \tag{5.28}$$

and putting $A_0 = 1$ (which implies no loss of generality, but simply fixes the magnitude of $E(r)$)

$$Z = \frac{1}{K\pi R^2}\left[\sum_{n=1}^{\infty} \frac{J_0^2(\gamma_n R)}{-i\mathcal{H}_n} A_n^2 + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} C_{\ell m} A_\ell A_m\right] \tag{5.29}$$

(5f) <u>Linear Equations for Mode Amplitudes and Simplified Form for Series Element</u>

Differentiate (5.29) with respect to the A_m, $m = 1, 2, \cdots \infty$ to give a set of linear equations for the A_m

$$\frac{J_0^2(\gamma_m R)}{-i\mathcal{H}_m} A_m + \sum_{\ell=0}^{\infty} C_{\ell m} A_\ell = 0 \quad , \quad m = 1, 2, \cdots \tag{5.30}$$

or

$$\begin{aligned} C_{01} + C_{11}A_1 + C_{21}A_2 + \cdots\cdots + \frac{J_0^2(\gamma_1 R)}{-i\mathcal{H}_1} A_1 &= 0 \\ C_{02} + C_{12}A_1 + C_{22}A_2 + \cdots\cdots + \frac{J_0^2(\gamma_2 R)}{-i\mathcal{H}_2} A_2 &= 0 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \end{aligned} \tag{5.31}$$

Putting (5.30) back into (5.29) gives

$$Z = \frac{1}{K\pi R^2}\left[\sum_{m=1}^{\infty} A_m \left(\frac{J_0^2(\gamma_m R)}{-i\mathcal{H}_m} A_m + \sum_{\ell=0}^{\infty} C_{\ell m} A_\ell\right) + \sum_{\ell=0}^{\infty} C_{\ell 0} A_\ell\right]$$

$$= \frac{1}{K\pi R^2} \sum_{\ell=0}^{\infty} C_{\ell 0} A_\ell = \frac{1}{K\pi R^2}\left[C_{00} + C_{10}A_1 + \cdots\right] \tag{5.32}$$

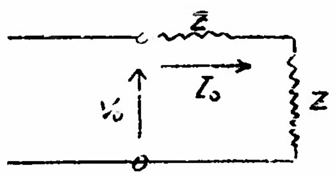
The equations (5.30) are an infinite set of linear equations for the coefficients of all the higher modes of E(r), which when solved and substituted in the variational form for Z, give by (5.32) a series for Z. The first approximation for Z is simply $Z = C_{00}/K\pi R^2$ and the second is obtained by solving for A_1 in (5.30) with $A_2, A_3, \cdots$ put $= 0$, giving

$$Z = \frac{1}{K\pi R^2}\left[C_{00} - \frac{C_{01}^2}{C_{11} + \frac{J_0^2(\gamma_1 R)}{-i\mathcal{H}_1}} + \cdots\right] \qquad (5.33)$$

This completes the formal solution of the problem and we are left with just the problem of numerical evaluation of the $C_{\ell m}$ which are then substituted in (5.39) and (5.32) or (5.33). In Section 6 the expressions are repeated in appropriate form and numerical evaluation made.

We note that after Z is found, the reflection coefficient $\mathcal{R}$ is obtained from transmission line theory as expressed in (4.20) - (4.23), but now we deal with the simple case $Z_{12} = \infty$, $Z_{11} - Z_{22} + 2Z_{12} = Z$

and $\frac{1+\mathcal{R}}{1-\mathcal{R}} = \frac{Z + Z'}{Z_0}$



where Z' is the termination of line II, as viewed from the junction.

(5g) Mode Amplitude Equations by Direct Matching

It is worth noting that the equations (5.30) for the A_m obtained from the condition that Z be stationary, also follow directly from the temperature matching equation (5.19).

To see this we expand the integrals of $E(r)$ using (5.6)

$$\int_0^R E(r) J_0(\gamma_n' r) r\,dr = K\sum_{\ell=0}^{\infty} A_\ell \int_0^R J_0(\gamma_\ell r) J_0(\gamma_n' r) r\,dr$$
$$= K\sum_{\ell=0}^{\infty} A_\ell D_{\ell n} \frac{RR'}{2}\left(\frac{K'}{K}\right)^{\frac{1}{2}} J_0(\gamma_n' R') \qquad (5.34)$$

Hence (5.19) becomes

$$Z = \frac{1}{K\pi R^2 A_0}\sum_{m=1}^{\infty}\frac{A_m J_0(\gamma_m R)}{-i\mathcal{H}_m} + \frac{1}{\pi(KK')^{1/2}RR'}\sum_{\ell=0}^{\infty} A_\ell \sum_{n=1}^{\infty}\frac{J_0(\gamma_n' r) D_{\ell n}}{-i\mathcal{H}_n' J_0(\gamma_n' R')} \qquad (5.35)$$

Multipy (5.35) by $J_0(\gamma_m r)\, r\, dr$ and integrate from 0 to R to give

$$0 = \frac{1}{KR^2}\frac{A_m}{(-i\mathcal{H}_m)}\frac{R^2}{2}J_0^2(\gamma_m R) + \frac{1}{(KK')^{1/2}}\sum_{l=0}^{\infty}A_l\sum_{n=1}^{\infty}\frac{D_{ln}D_{mn}}{-i\mathcal{H}_n'}\frac{1}{2}\left(\frac{K}{K'}\right)^{\frac{1}{2}}$$

$$= \frac{J_0^2(\gamma_m R)}{-i\mathcal{H}_m}\frac{A_m}{2K} + \frac{1}{2K}\sum_{l=0}^{\infty}A_l C_{lm}$$

which agrees with (5.30).

Thus we check the algebra leading to (5.30) and prove again that Z is stationary.

6. NUMERICAL EVALUATION OF THE SERIES ELEMENT, Z, AND REFLECTION COEFFICIENT $\mathcal{R}$

(6a) Summary of Procedures

The computation of the equivalent series element Z for the junction of the two rods is still a complicated matter, since the Bessel function summations, which give the $C_{\ell m}$ must first be evaluated. These sums all converge for large n like $(1/n)^3$, but may be slower for a practicable number of terms. Accordingly the convergence is assisted by summation formulas for the series of asymptotic forms of the Bessel functions. The difference series may then be more quickly summed.

The formulas for Z and the various series involved are first tabulated and put into dimensionless form. The asymptotic forms of the Bessel functions in C_{00} are introduced, giving a Fourier series as an approximation to Z. A numerical part of this series is summed by use of the Ψ function, its derivatives and asymptotic form, and the result is checked by a second method, using the Euler-Maclaurin formula. The Fourier series is summed by starting with simpler series with easily obtained sums, and then building up the desired series by successive integration, leading to rapidly convergent power series. Two forms of the power series are given with different ranges of convergence. Numerical results obtained from these formulas are discussed, tabulated (in Appendix IV) and plotted for a range of γ from 1 to 10 and $\mathcal{K}'R'$ from 0 to $1.6(1+i)$. The values are intended as a survey of the physical behavior and as a test of the formulas, hence are not pushed to high accuracy, but are based mainly on the first term of the variational expression for Z depending just on C_{00}. The basic terms of the Bessel function series are given, from which the first order and all higher order calculations would start. Sufficient data are given so that Z may be simply calculated to first approximation in the general case of change of cross section and material (four parameters). (Z/Z_0) and $\mathcal{R}$ (for matched termination) are themselves tabulated for just the change of cross section (two parameters) over the parameter ranges given above. The second approximation is evaluated for various values of γ and $\mathcal{K}'R'$, and makes at most a 10% correction to (Z/Z_0) (2% to $\mathcal{R}$).

Procedures for evaluation of the $C_{\ell m}$ for any values of ℓ and m are now considered, since these must be known to obtain the higher order terms in Z. Again the asymptotic (Fourier) series is obtained and summed by procedures similar to those used on C_{00}. The general sum for $C_{0\ell}$ is expressed in terms of the Ψ function, trigonometric functions and a definite integral. Two procedures for calculating the latter are given in Appendix II, namely series expansion and numerical integration by Filon's formula. Special consideration must be given to the occurrence of an accidental singularity in one of the terms of the series due to the use of the asymptotic forms; it is proved, however, that the exact series is always finite. The series of the coefficients $C_{\ell\ell}$ is summed by differentiating the series for $C_{0\ell}$. The quantities $C_{\ell m}$ are easily obtained from $C_{0\ell}$ and C_{0m} by a simple formula. It is noted that the Hahn functions are easily summed by the methods of this section and formulas for them are given. Finally the additional factor in the sums which depends on frequency is discussed. This factor is easily taken account of numerically when the wavelength is long (compared to rod radii), but is more important as we go further from the static limit. Summation procedure for series containing the next term in the expansion of this factor is indicated.

(6b) Recapitulation of Formulas

We have
$$Z = \frac{1}{K\pi R^2}\sum_{\ell=0}^{\infty} C_{\ell 0}A_\ell \cong \frac{1}{K\pi R^2}\left[C_{00} - \frac{C_{01}^2}{C_{11} + \frac{J_0^2(\gamma_1 R)}{-i\mathcal{K}_1}}\right] \tag{6.1}$$

$$C_{\ell m} = \sum_{n=1}^{\infty} D_{\ell n}D_{mn}/(-i\mathcal{K}_n') \tag{6.2}$$

$$D_{\ell n} = 2\left(\frac{K}{K'}\right)^{\frac{1}{2}}\frac{\beta_n}{\beta_n^2-\gamma^2\beta_\ell^2}\;\frac{J_0(\beta_\ell)J_1(\beta_n/\gamma)}{J_0(\beta_n)} \tag{6.3}$$

where

$$\gamma_n R = \gamma_n' R' = \beta_n \; ; \quad J_1(\beta_n) = 0 \quad , \quad \delta = R'/R . \tag{6.4}$$

Thus

$$D_{on} = 2\left(\frac{K}{K'}\right)^{\frac{1}{2}} \frac{J_1(\beta_n/\delta)}{\beta_n J_0(\beta_n)} \quad , \quad D_{in} = 2\left(\frac{K}{K'}\right)^{\frac{1}{2}} \frac{\beta_n}{\beta_n^2 - \delta^2 \beta_1^2} \; \frac{J_0(\beta_1) J_1(\beta_n/\delta)}{J_1(\beta_n)} \tag{6.5}$$

$$C_{oo} = \frac{4K}{K'} \sum_{n=1}^{\infty} \frac{1}{-i\mathcal{H}_n'} \frac{J_1^2(\beta_n/\delta)}{J_0^2(\beta_n)} = \frac{4KR'}{K'} \sum_{n=1}^{\infty} \frac{J_1^2(\beta_n/\delta)}{\beta_n^3 J_0^2(\beta_n)} \frac{1}{\left[1 - \left(\frac{\mathcal{H}'R'}{\beta_n}\right)^2\right]^{\frac{1}{2}}} \tag{6.6}$$

since

$$-i\mathcal{H}_n' = \left[\gamma_n'^2 - \mathcal{H}'^2\right]^{\frac{1}{2}} = \frac{\beta_n}{R'}\left[1 - \frac{\mathcal{H}'^2 R'^2}{\beta_n^2}\right]^{\frac{1}{2}} \quad ; \quad \mathcal{H}' = (1+i)\sqrt{\frac{\omega}{2D}} \tag{6.7}$$

where the roots with positive real parts are taken in (6.7), so that the wave $e^{i\mathcal{H}_n z - i\omega t}$ attenuates in the propagation direction, namely toward $+z$.

Thus the first approximation to Z is

$$Z \simeq \frac{C_{oo}}{K\pi R^2} = \frac{4R'}{K'\pi R^2} \sum_{n=0}^{\infty} \frac{J_1^2(\beta_n/\delta)}{\beta_n^3 J_0^2(\beta_n)\left[1 - \left(\frac{\mathcal{H}'R'}{\beta_n}\right)^2\right]^{\frac{1}{2}}} \tag{6.8}$$

We proceed now to the numerical evaluation of (6.8), which is a series decreasing as $1/n^3$, hence some 20 terms could be required for 4 place accuracy.

(6c) Introduction of Asymptotic Forms

The convergence of (6.8)can be improved, however, by subtracting the asymptotic form of the general term and summing analytically the series of asymptotic terms. We have the asymptotic expansions

$$\left.\begin{aligned} J_0(x) &\simeq \sqrt{\frac{2}{\pi x}}\left[\left(1 - \frac{9}{2(8x)^2} + \frac{9\times25\times49}{24(8x)^4} - \cdots\right)\cos\phi_0 - \left(-\frac{1}{8x} + \frac{9\times25}{6(8x)^3} \cdots\right)\sin\phi_0\right] \\ J_1(x) &\simeq \sqrt{\frac{2}{\pi x}}\left[\left(1 + \frac{3\times5}{2(8x)^2} - \frac{3\times5\times21\times45}{24(8x)^4} \cdots\right)\cos\phi_1 - \left(\frac{3}{8x} - \frac{3\times5\times21}{6(8x)^3} \cdots\right)\sin\phi_1\right] \end{aligned}\right\} \tag{6.9}$$

$$\phi_0 = x - \pi/4 \quad , \quad \phi_1 = x - 3\pi/4 \tag{6.10}$$

$$\beta_n = \pi\left(n + \frac{1}{4} - \frac{.151982}{4n+1} + \frac{.015399}{(4n+1)^3} - \frac{.245270}{(4n+1)^5} \cdots\cdots\right)$$

Taking the leading terms in (6.9) we have

$$S_n = \frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \simeq \frac{\gamma\cos^2\left(\frac{\pi(n+1/4)}{\gamma} - \frac{3\pi}{4}\right)}{\pi^3(n+1/4)^3\cos^2(\pi[n+1/4]-\pi/4)} = \frac{\gamma\left(1+\cos\left[\frac{2\pi}{\gamma}(n+1/4)-3\pi/2\right]\right)}{2\pi^3(n+1/4)^3}$$

$$= \frac{\gamma}{2\pi^3(n+1/4)^3} - \frac{\gamma\sin\frac{2\pi}{\gamma}(n+1/4)}{2\pi^3(n+1/4)^3} \tag{6.11}$$

Hence

$$S = \sum_{n=1}^{\infty}\frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \simeq \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{1}{(n+1/4)^3} - \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{2\pi}{\gamma}(n+1/4)}{(n+1/4)^3} \tag{6.12}$$

(6d) **Evaluation of Numerical Series by Use of the Ψ Function.**

The first sum

$$\sum_{n=1}^{\infty}\frac{1}{(n+1/4)^3} = -\frac{1}{2}\Psi''(1/4) \tag{6.13}$$

where $\Psi(z) = \frac{d\ln z!}{dz} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) - C_E$ (See Jahnke and Ende[7], p. 18)

$$\simeq \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \cdots \tag{6.14}$$

$C_E = .577216 =$ Euler's constant

$$\Psi'(z) = \sum_{n=1}^{\infty}\frac{1}{(n+z)^2} \simeq \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} \tag{6.15}$$

$$\Psi''(z) = -2\sum_{n=1}^{\infty}\frac{1}{(n+z)^3} \simeq -\frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} \cdots \tag{6.16}$$

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(7) E. Jahnke and F. Emde "Tables of Functions, 3rd Edition, 1933

The function $\Psi''(z)$ is tabulated (Davis (8)) but can easily be calculated from the last series in (6.16) The term $1/6z^6$ is negligible if $z \geq 5$, hence put

$$\sum_{n=1}^{\infty} \frac{1}{(n+1/4)^3} = \sum_{n=1}^{5} \frac{1}{(n+1/4)^3} + \sum_{n'=n-5=1}^{\infty} \frac{1}{(n'+5\frac{1}{4})^3}$$

$$= .64886 - \frac{1}{2}\Psi''(5\tfrac{1}{4}) = .66387 \qquad (6.17)$$

[Note that Davis uses and tabulates the function $\Psi(z-1) = \dfrac{d \ln \Gamma(z)}{dz}$]

(6e) <u>Check by Euler-Maclaurin Formula</u>

An alternative method which will check the result (6.17) makes use of the Euler-Maclaurin formula (Jeffreys and Jeffreys[9] p. 279)

$$\sum_{l=1}^{n} f(n) = \int_0^n f(x)\,dx + \tfrac{1}{2}(f(n)-f(0)) + b_2(f'(n)-f'(0)) + b_4(f'''(n)-f'''(0)) \cdots \qquad (6.18)$$

$$b_{2r} = (-)^{r-1} B_r/(2r)! \quad ; \quad b_2 = \frac{1}{12} \quad ; \quad b_4 = -\frac{1}{30 \times 24} \quad ; \quad b_6 = \frac{1}{42 \times 720}, \cdots$$

where B_r are the Bernoulli numbers (as given in Dwight[10] p. 9 for example, but differing from the definition used in Jeffreys and Jeffreys)

Now consider

$$\sum_{n'=1}^{\infty} \frac{1}{(n'+5\frac{1}{4})^3} = \int_0^{\infty} \frac{dx}{(x+5\frac{1}{4})^3} + \frac{1}{2}\left(0 - \frac{1}{(5\frac{1}{4})^3}\right) + \frac{1}{12}\left(0 + \frac{3}{(5\frac{1}{4})^4}\right) - \frac{1}{720}\left(0 - \frac{60}{(5\frac{1}{4})^6}\right) \cdots \qquad (6.19)$$

$$= .01814 - .00346 + .00033 - .00000 = .01501$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(n+1/4)^3} = .64886 + .01501 = .66387$$

which checks (6.17)

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(8) - H. T. Davis - "Tables of the Higher Mathematical Functions" Vol. I, 1933, Vol. II, 1935, The Principia Press.

(9) - H. Jeffreys and B. Jeffreys - "Methods of Mathematical Physics", Cambridge University Press, 2nd Edition, 1950.

(10)- H. B. Dwight - Tables of Integrals and other Mathematical Data - The MacMillan Co. 1934

(6f) Summation of Fourier Series of Asymptotic Terms, First Form for $\gamma > 3$

The evaluation of the second term in (6.12), namely $\sum_{n=1}^{\infty} \frac{\sin \frac{2\pi}{\gamma}(n+1/4)}{(n+1/4)^3}$ requires more elaborate analysis. We note that if

$$f(z) = \sum_{n=1}^{\infty} \frac{\sin(n+1/4)z}{(n+1/4)^3} \quad ; \tag{6.20}$$

then

$$f'(z) = \sum_{n=1}^{\infty} \frac{\cos(n+1/4)z}{(n+1/4)^2} \tag{6.21}$$

$$f''(z) = -\sum_{n=1}^{\infty} \sin(n+1/4)z/(n+1/4) \tag{6.22}$$

$$f'''(z) = -\sum_{n=1}^{\infty} \cos(n+1/4)z = -\cos\frac{z}{4}\left(\sum_{n=1}^{\infty} \cos nz\right) + \sin\frac{z}{4}\left(\sum_{n=1}^{\infty} \sin nz\right) \tag{6.23}$$

The series in (6.23) are not convergent in the simple definition of convergence but rather oscillate. However, they may be summed by a slight extension of the usual definition, and then integrated, yielding in (6.22), (6.21) and (6.20) series convergent in the ordinary sense.

$$\text{Thus} \quad \sum_{n=0}^{\infty} e^{inz} = \frac{1}{1-e^{iz}} = \frac{1}{2} + \frac{i}{2}\cot\frac{z}{2} \tag{6.24}$$

or, formally, we may write (using the symbol $(=)$ to mean "is formally equal to")

$$\sum_{n=0}^{\infty} \cos nz \,(=)\, \frac{1}{2} \quad ; \quad \sum_{n=0}^{\infty} \sin nz \,(=)\, \frac{1}{2}\cot\frac{z}{2} \tag{6.25}$$

Note that (6.24) is convergent if z has a small positive imaginary part, although this is not true for (6.25). The meaning of (6.25) will not be made precise here, but we shall merely use the formal equalities to suggest new equalities obtained by operations such as integration. The new equalities can then be checked and established independently. Regardless of the way (6.25) is given a precise meaning, the equality clearly cannot hold at $z=0$ or $z=2\pi$ and may be expected to hold only over $0<z<2\pi$. Then putting (6.25) in (6.23) will give, in the same formal sense, (note the change of sign and summation limit),

$$\sum_{n=0}^{\infty} \cos(n+1/4)z \,(=)\, \frac{1}{2}\cos\frac{z}{4} - \sin\frac{z}{4}\left(\frac{1}{2}\cot\frac{z}{2}\right) = \left(\tfrac{1}{4}\cos\tfrac{z}{4}\right); \quad 0<z<2\pi . \tag{6.26}$$

Putting $x = z/4$, (6.26) becomes

$$\sum_{n=0}^{\infty} \cos(4n+1)x \,(=)\, \frac{\sec x}{4} \qquad 0<x<\pi/2 \tag{6.27}$$

$$(=)\, \frac{1}{4}\sum_{n=0}^{\infty} \frac{E_n x^{2n}}{(2n)!} \quad (\text{converges for } |x|<\pi/2) \tag{6.28}$$

where E_n = the Euler numbers (See Dwight[10] p.9)

$$E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385,\ E_5 = 50521,\ E_6 = 2{,}702{,}765,\ E_7 = 199360981 \cdots\cdots \tag{6.29}$$

Integrate (6.28) between $\pi/4$ and x, noting $\sin(4n+1)\frac{\pi}{4} = (-)^n \frac{\sqrt{2}}{2}$

$$\sum_{n=0}^{\infty} \frac{\sin(4n+1)x}{4n+1} - \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-)^n}{4n+1} = \frac{1}{4}\ln(\sec x + \tan x) - \frac{1}{4}\ln(\sqrt{2}+1) \tag{6.30}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{E_n x^{2n+1}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{E_n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} \tag{6.31}$$

Now

$$\sum_{n=0}^{\infty} \frac{(-)^n}{4n+1} = \int_0^1 dt \sum_{n=0}^{\infty} (-1)^n t^{4n} = \int_0^1 \frac{dt}{1+t^4}$$

$$= \left[\frac{1}{4\sqrt{2}} \ln \frac{t^2+t\sqrt{2}+1}{t^2-t\sqrt{2}+1} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{t\sqrt{2}}{1-t^2}\right]_0^1 = \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} \ln \frac{2+\sqrt{2}}{2-\sqrt{2}} \tag{6.32}$$

Putting (6.32) into (6.30), (6.31) gives

$$\sum_{n=0}^{\infty} \frac{\sin(4n+1)x}{4n+1} = \frac{\pi}{8} + \frac{1}{4}\ln(\sec x + \tan x) \tag{6.33}$$

$$= \frac{\pi}{8} + \frac{1}{4}\sum_{n=0}^{\infty} E_n \frac{x^{2n+1}}{(2n+1)!} \tag{6.34}$$

Evidently (6.33) and (6.34) do not hold $x=0$ or $\pi/2$; but the series on the left converges in the usual sense elsewhere. Integrating (6.34) over x between indefinite limits gives

$$\sum_{n=0}^{\infty} \frac{\cos(4n+1)x}{(4n+1)^2} = A - \frac{\pi x}{8} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{E_n}{(2n+2)!} x^{2n+2} \tag{6.35}$$

where A is a constant to be determined. Since (6.33) is uniformly convergent in any interval $0 < \delta \leq x \leq \pi/2 - \delta < \pi/2$, the integrated form (6.35) holds in any such interval. Also (6.33) converges at $x = 0$ and has only a logarithmic singularity at $\pi/2$, hence by making δ small enough the contributions to the integration of (6.33) over x of the intervals 0 to δ, and $\pi/2 - \delta$ to $\pi/2$ can be made negligible, and we expect (6.35) to hold at x=0, and $\pi/2$.

Taking x=0 in (6.35) requires that

$$A = \sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} = 1 + \frac{1}{16}\Psi'\left(\frac{1}{4}\right) = 1.07483 \tag{6.36}$$

As a check we take $x = \pi/4$; $\cos(4n+1)\frac{\pi}{4} = (-)^n\sqrt{2}/2$, and now we must have

$$A = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-)^n}{(4n+1)^2} + \frac{\pi^2}{32} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{E_n}{(2n+2)!}\left(\frac{\pi^2}{16}\right)^{n+1}$$

$$= \frac{\sqrt{2}}{2}\left[1 + \frac{1}{64}\Psi'\left(\frac{1}{8}\right) - \frac{1}{25} - \frac{1}{64}\Psi'\left(\frac{5}{8}\right)\right] + \frac{\pi^2}{32} + \frac{1}{4}\times .32617 = 1.0748$$

which checks (6.36). Thus we have reason to believe (6.35) with (6.36) for A is a correct equality.

Finally integrating (6.35) from 0 to x, (note (6.35) now converges at $x=0$) gives

$$\sum_{n=0}^{\infty}\frac{\sin(4n+1)x}{(4n+1)^3} = 1.07483\,x - \frac{\pi x^2}{16} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{E_n}{(2n+3)!}x^{2n+3} \qquad (6.37)$$

To check (6.37), take $x = \pi/4$, and put $\sin(4n+1)\pi/4 = (-)^n\sqrt{2}/2$, so that the sum becomes

$$\frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-)^n}{(4n+1)^3} = \frac{\sqrt{2}}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(8n+1)^3} - \sum_{n=0}^{\infty}\frac{1}{(8n+5)^3}\right)$$

$$= \frac{\sqrt{2}}{2}\times\frac{1}{512}\times\left(-\frac{1}{2}\Psi''\left(-\frac{7}{8}\right) + \frac{1}{2}\Psi''\left(-\frac{3}{8}\right)\right) = .70219 \qquad (6.38)$$

to be compared with 1.07483x .78540 - .19635x .61685 - 1/4 x .08344 = .70219 directly from (6.37). The coefficients in (6.37) $E_n/(2n+3)!$ are tabulated below (and in Appendix IV).

n	$E_n/(2n+3)!$
0	.166 666 667
1	.008 333 333
2	.000 992 064
3	.000 168 100
4	.000 034 697
5	.000 008 113
6	.000 002 067
7	.000 000 561

(6g) Second form of Summation of Fourier Series for γ Near 1

The series (6.37) for $\tilde{S}$ (the asymptotic series to S) does not converge well for $\pi/2 > \gamma > 1$, or $\gamma < \pi/2 = 1.5708$ (thus the eighth term is $\frac{E_7}{7!}(\frac{\pi}{2})^7 \cong .001$). To cover the range $1 \le \gamma \le \pi/2$ another series is useful which may be derived similarly, starting from (6.27), as follows:

$$\sum_{n=0}^{\infty} \cos(4n+1)x \,(=)\, \frac{\sec x}{4} = \frac{\csc y}{4} = \frac{1}{4y} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{(2^{2n}-2)}{2n!} B_n y^{2n-1}; \quad y = \frac{\pi}{2} - x . \qquad (6.39)$$

Integrating from $x = \pi/4$ to x or from $y = \pi/4$ to y as in (6.30), and using (6.32)

$$\sum_{n=0}^{\infty} \frac{\sin(4n+1)x}{4n+1} - \frac{\sqrt{2}}{2}\left[\frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} \ln\frac{2+\sqrt{2}}{2-\sqrt{2}}\right] = \frac{1}{4} \ln\tan\frac{y}{2}\Big]_y^{\pi/4} = \frac{1}{4}\ln(\sqrt{2}-1) - \frac{1}{4}\ln\tan\frac{y}{2}$$

$$= \frac{1}{4}\left(\ln\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n!}\frac{B_n}{2n}\left(\frac{\pi}{4}\right)^{2n} - \ln y - \sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n!}\frac{B_n}{2n} y^{2n}\right). \qquad (6.40)$$

Note that $\ln\tan\frac{y}{2} = \ln y - \ln 2 + y^2/12 - - - - -$, hence the two forms of (6.40) give

$$\frac{1}{4}\left[\ln(\sqrt{2}-1) + \ln 2 - \ln y - y^2/12 \cdots\right] = \frac{1}{4}\left[\ln\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n!}\frac{B_n}{2n}\left(\frac{\pi}{4}\right)^{2n} - \ln y - y^2/12 \cdots\right] \qquad (6.41)$$

Thus the numerical series in (6.41) or (6.40) is evaluated and (6.40) becomes, after cancellations,

$$\sum_{n=0}^{\infty}\frac{\sin(4n+1)x}{4n+1} = \frac{\pi}{8} + \frac{\ln 2}{4} - \frac{\ln y}{4} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n)!} B_n y^{2n}. \qquad (6.42)$$

Integrate (6.42) from x to $\pi/2$ or from y to 0, noting that the singularity at $y=0$ is integrable, to give

$$\sum_{n=0}^{\infty}\frac{\cos(4n+1)x}{(4n+1)^2} = \left(\frac{\pi}{8} + \frac{\ln 2}{4}\right)y - \frac{1}{4}(y\ln y - y) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n+1)!} B_n y^{2n+1}. \qquad (6.43)$$

Integrating (6.43) over the same ranges, using (6.17) gives

$$\sum_{n=0}^{\infty}\frac{\sin(4n+1)x}{(4n+1)^3} = 1 + \frac{.66387}{4} - \left(\frac{\pi}{8} + \frac{\ln 2}{4}\right)\frac{y^2}{2} + \frac{1}{4}\left(\frac{y^2\ln y}{2} - \frac{3y^2}{4}\right) + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n+2)!} B_n y^{2n+2} \qquad (6.44)$$

or

$$\sum_{n=0}^{\infty}\frac{\sin(4n+1)x}{(4n+1)^3} = 1.010373 - .470493\, y^2 + .125000\, y^2 \ln y + .250000 \sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n+2)!} B_n y^{2n+2} \qquad (6.45)$$

To check (6.45), take $x = y = \pi/4$, and compare with (6.38), giving in both cases, 0.70219.

The coefficients in the series in (6.45) are as follows:(also given in Appendix IV)

n	$\frac{2^{2n}-2}{2n(2n+2)!}B_n$
1	.006 944 444
2	.000 162 037
3	.000 006 102
4	.000 000 292
5	.000 000 016
6	.000 000 001

These are substantially smaller than the coefficients in (6.37), a result probably due to explicit removal of the logarithmic singularity at $x = \pi/2$ or y=0 in (6.44).

Both series (6.37) and (6.46) converge over the entire physical range of γ, but (6.37) diverges at $x = \pi/2$ or $\gamma = 1$, while (6.46) diverges only when y reaches π; the largest value of physical interest is $y = \pi/2$ at $\gamma = \infty$.
Thus (6.46) is more rapidly convergent for $1 \le \gamma \le 3, (0 \le y \le (\pi/2))$ while (6.37) is better for $3 \le \gamma \le \infty ((\pi/6) \ge x \ge 0)$.

(6h) Evaluation of Z and R, General Formulas and Physical Parameters

We can now proceed with the calculation of Z, or better, of Z/Z_o.
From (6.8), and (5.15)

$$\frac{Z}{Z_o} \cong -i\mathcal{K}C_{oo} = -4i\mathcal{K}R'\frac{K}{K'}\sum_{n=1}^{\infty}\frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_o^2(\beta_n)[1-(\mathcal{K}'R'/\beta_n)^2]^{1/2}} \qquad (6.46)$$

$$\mathcal{K} = (1+i)\sqrt{\frac{\omega}{2D}}, \qquad \mathcal{K}' = (1+i)\sqrt{\frac{\omega}{2D'}}$$

$$\gamma = R'/R \quad ; \quad J_1(\beta_n) = 0$$

Thus Z/Z_o depends on four parameters, $(K/K'), \gamma, \mathcal{K}R, \mathcal{K}'R'$

$$Z/Z_o = -4i\mathcal{K}R\gamma(K/K')\mathcal{S} \; ; \; \mathcal{S} = \sum_{n=1}^{\infty}\frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_o^2(\beta_n)[1-(\mathcal{K}'R'/\beta_n)^2]^{1/2}} = \sum_{n=1}^{\infty} S_n\left[1-\left(\frac{\mathcal{K}'R'}{\beta_n}\right)^2\right]^{-\frac{1}{2}} \qquad (6.47)$$

Now $\mathcal{S}$ is asymptotic to S, given by $S = \sum_{n=1}^{\infty}\left(J_1^2(\beta_n/\gamma)/\beta_n^3 J_o^2(\beta_n)\right) = \sum_{n=1}^{\infty} S_n \qquad (6.48)$

which in turn is asymptotic to

$$\tilde{S} = \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{1}{(n+1/4)^3} - \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2\pi/\gamma)(n+1/4)}{(n+1/4)^3} \qquad (6.49)$$

$$= \frac{.016816}{x} - 1.74245 + .31831x + 1.62114\frac{\sin x}{x} + .40528\sum_{n=0}^{\infty}\frac{E_n}{(2n+3)!}x^{2n+2}, \quad x = \frac{\pi}{2\gamma} \qquad (6.50)$$

by (6.37).

From (6.45), however,

$$\tilde{\zeta} = \gamma\left[-1.032049\left(1-\sin\frac{\pi}{2\gamma}\right) + .485572\, y^3 - .129006\, y^2 \ln y - .258012 \sum_{n=1}^{\infty} \frac{(2^{2n}-2)}{2n(2n+2)!} B_n\, y^{2n+2}\right] \quad (6.51)$$

where $y = \pi/2 - \pi/2\gamma = \pi/2 - x$.

Thus $\mathcal{S}$ depends on just the two parameters γ and $\kappa' R'$, while S or $\tilde{S}$ depend only on γ. This makes it possible to tabulate $\mathcal{S}$ as a function of γ and $\kappa' R'$ and permits evaluation of Z/Z_0 and the reflection coefficient for all changes of cross section and of material by a simple calculation (illustrated in subsection 61) which brings in the two additional parameters K/K' and κ R. However, if there is no change of material but only a change of cross section, then $\kappa = \kappa'$, K=K', and Z/Z_0 depends only on γ and $\kappa' R'$ (or κR) and the reflection coefficient may be tabulated directly as a function of these two parameters (Appendix IV).

These considerations make use thus far of just the first approximation (6.46) to Z/Z_0 in the variational form. The effects of higher order terms will be considered later, but in general will amount to small corrections.

The voltage (or temperature) reflection coefficient for semi-infinite termination of the rod R' is then given by (4.19), (4.20) or (4.23), and takes the forms, for the special case of a single series element Z at the junction,

$$\frac{1+\mathcal{R}}{1-\mathcal{R}} = \frac{Z}{Z_0} + \frac{Z_0'}{Z_0} = \frac{Z}{Z_0} + \frac{K}{K'}\,\frac{\kappa R^2}{\kappa' R'^2}. \quad (6.52)$$

We note the second term is real and for a change of cross section without change of material is just $(1/\gamma^2)$.

(61) Numerical Results

Some calculations have been made, based on the formulas of this Section, in order to survey the behavior of the series impedance representing the junction over a reasonable range of the physical parameters. In addition the calculations serve to test the usefulness of the formulas for numerical purposes. In view of the large amount of numerical work required for a complete and accurate tabulation of the impedance as a function of four parameters in the general case, and since this is intended as a survey of the general features of the problem, the work has not been pushed to high accuracy, nor has it been carefully checked. However, enough has been done both to give these general features and to serve as a starting point for an accurate result for any particular set of parameters.

Accordingly, most attention has been given to calculations using just the first term of the variational expression for Z, as in (6.46), but at a few points the magnitude of the second term, as in (6.1), has been found, to give some idea of the accuracy of the first approximation. In Appendix IV a tabulation is given of the terms $S_n(\gamma)$ defined in (6.48), over the range of γ from 1 to 10, and also of the terms of the asymptotic series $\tilde{S}_n$ of (6.49), to illustrate the summation procedure. The $S_n(\gamma)$ are the basic quantities entering all the numerical series for the coefficients $C_{\ell m}$ and form the starting

point for any more elaborate or precise calculation. From them the sums $\mathcal{L}(\sigma, \varkappa'R')$ (or $\mathcal{L}^{(0,0)}(\sigma, \varkappa'R')$) are obtained for the same range of σ values, and for a range of $\varkappa'R'$ from 0 to 1.6 $(1+i)$ corresponding to a wave length comparable to the rod diameter. [Since $\varkappa' = \sqrt{\nu/2D}\,(1+i)$ the maximum value of $\varkappa'R'$ corresponds to a maximum frequency of $\nu \sim D/R'^2$, which at low temperatures, for metals, and for reasonably sized rods, is at least a few thousand cycles per second. This is about as high a frequency as the thermometers in use can respond to effectively, although further progress here might extend the range of interest of $\varkappa'R'$].

From $\mathcal{L}^{(0,0)}(\sigma, \varkappa'R')$, by (6.46), $Z/Z_0(\sigma, \varkappa'R', \varkappa R, K/K')$ can be easily found for any values of the four variables. If, however, we take the special case in which no change of material occurs, then $\varkappa R = \varkappa'R'/\sigma$ and $(K/K') = 1$, and (Z/Z_0) becomes a function of just two variables. (Z/Z_0) is tabulated in the next table for this case, over the same range of σ and $\varkappa'R'$ as used for $\mathcal{L}^{(0,0)}$, and also the corresponding reflection coefficient, $\mathcal{R}$, for a wave incident on the junction from the smaller rod, when the larger rod has a matched termination. The results are plotted and show that (Z/Z_0) varies from 0 at $\sigma = 1$, through a maximum at about $\sigma = 2.5$, to smaller values as σ increases to 10. Z remains always in the first quadrant, as required by the theorem of Section 4, and in fact is nearly real. (Z/Z_0) is quite frequency dependent, rising from 0 at zero frequency. However $\mathcal{L}^{(0,0)}$ is much less frequency sensitive, as shown in the plot, and varies by less than 3 1/2% over $\varkappa'R' = 0$ to 1.6 $(1+i)$ (in the real part which is the dominantpart).

The values of $\mathcal{R}$ rise from 0 at $\sigma = 1$ rapidly to nearly -1 as σ increases to 10 (the large value of σ corresponds to a short circuit, i.e. an end held at fixed T by the large mass of the second rod), and also varies substantially with frequency in both magnitude (which decreases at higher frequencies) and phase. Thus at $\sigma = 2.5$, the reflection at $\varkappa'R' = 1.6(1+i)$ is down from .7 to .4, and varies nearly linearly with $\varkappa'R'$.

Finally the second approximation to (Z/Z_0) is obtained for selected values of σ and $\varkappa'R'$. This requires tabulation of $\mathcal{L}^{(0,0)}(\sigma, \varkappa'R')$ and $\mathcal{L}^{(1,1)}(\sigma, \varkappa'R')$ as defined in (6.54) and (Z/Z_0) can then be calculated for any values of the four parameters to second approximation. The appropriate formula, obtained by transformation of (6.1), using (6.53), (6.54) is

$$\frac{Z}{Z_0} = -4i\,\varkappa R\frac{K}{K'}\left[\mathcal{L}^{(0,0)} - \frac{(\mathcal{L}^{(0,1)})^2}{\mathcal{L}^{(1,1)} + \dfrac{K'}{4K\sigma\beta_1\left[1 - \dfrac{\varkappa^2R^2}{\beta_1^2}\right]^{1/2}}}\right].$$

In the same special case as above, i.e. no change of material, (Z/Z_0) and $\mathcal{R}$ are tabulated explicitly for the selected values of σ and $\varkappa'R'$. The results show the second term in (Z/Z_0) is at most about 10% of the first term, being smaller at larger values of σ and lower frequencies. In $|\mathcal{R}|$ the corrections are at most only about 2%.

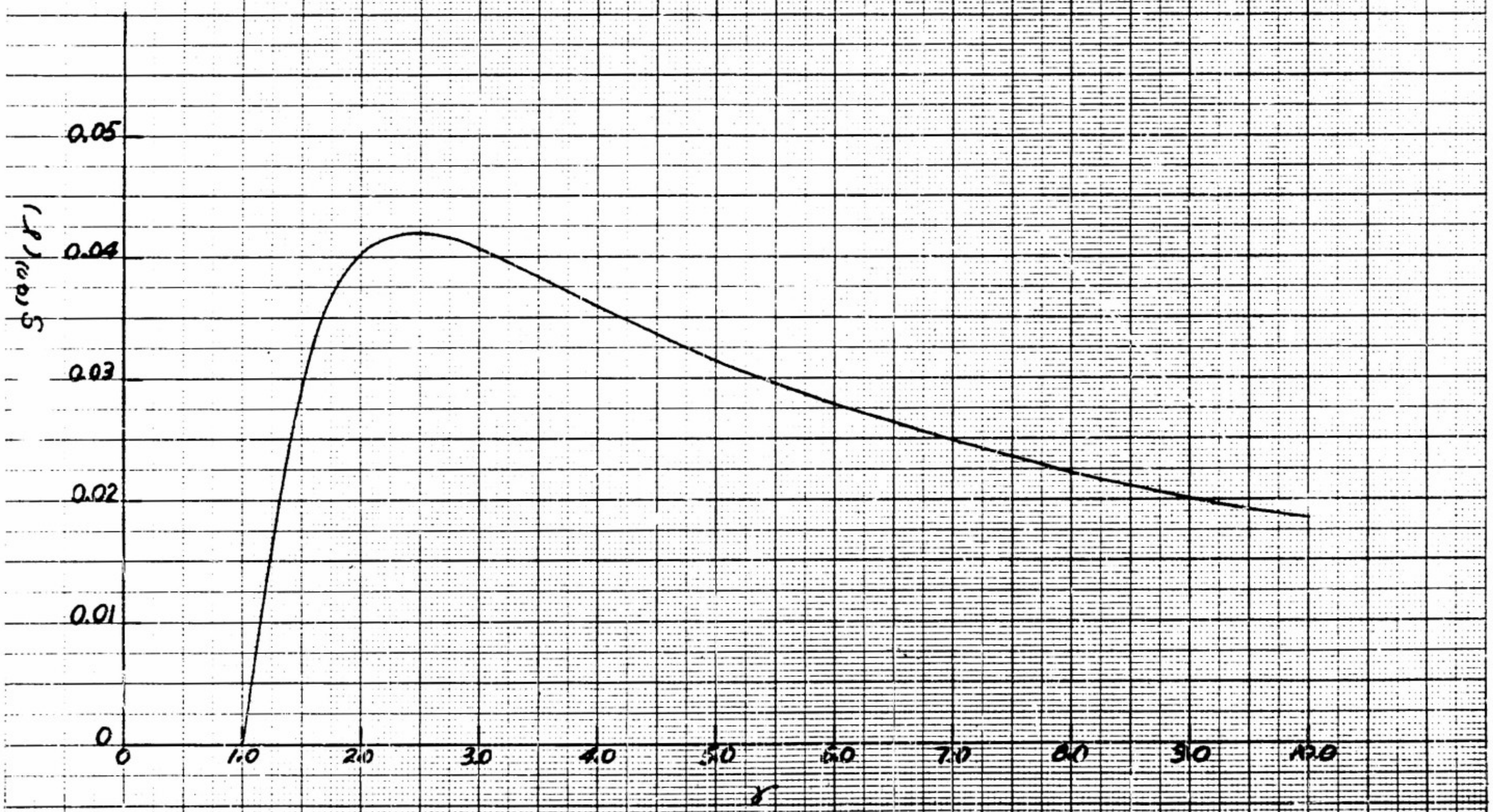
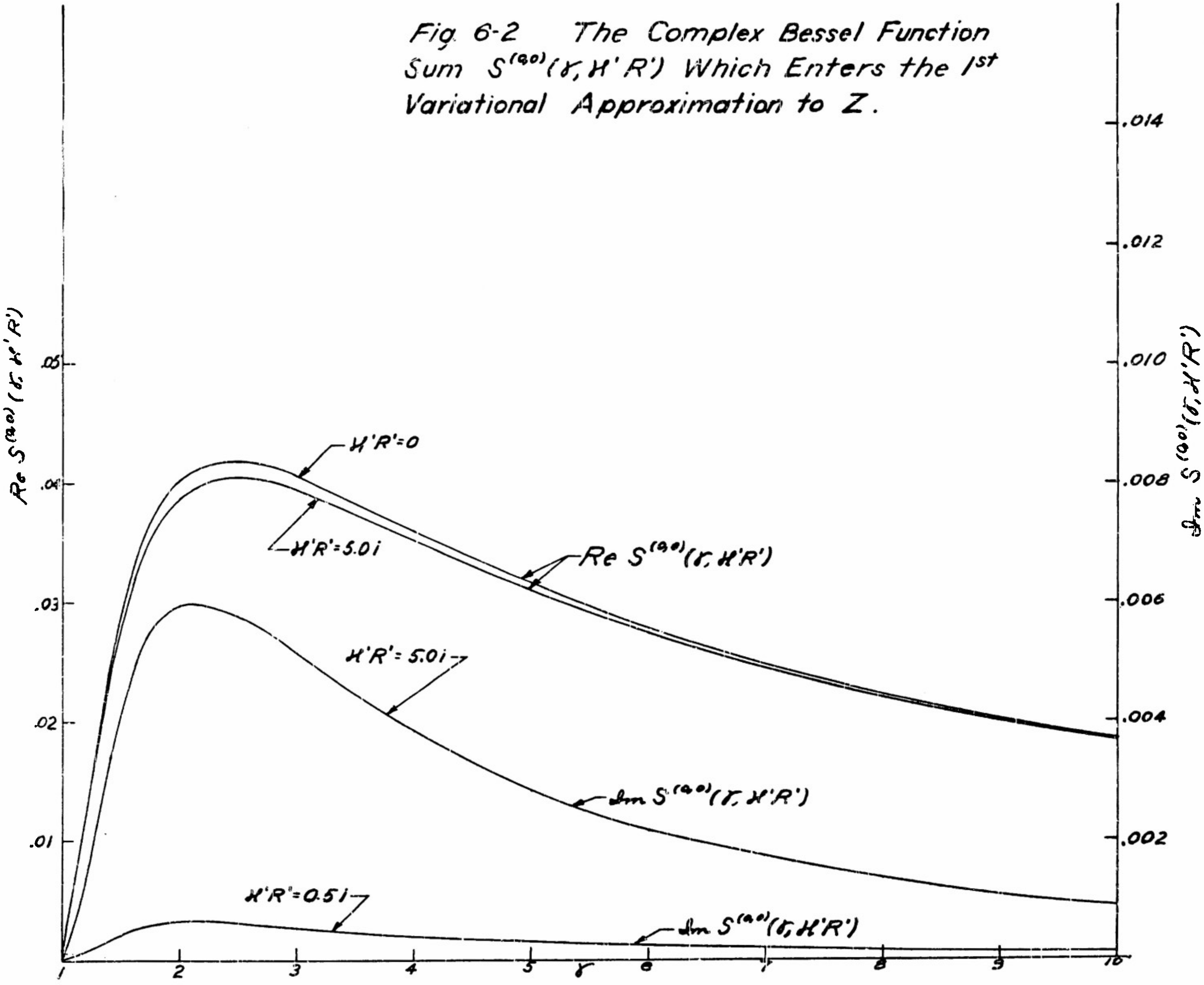


Fig. 6-1

Basic Bessel Function Sum $S^{(00)}(\gamma)$

Fig. 6-2 The Complex Bessel Function Sum $S^{(a0)}(\gamma, \varkappa' R')$ Which Enters the 1st Variational Approximation to Z.



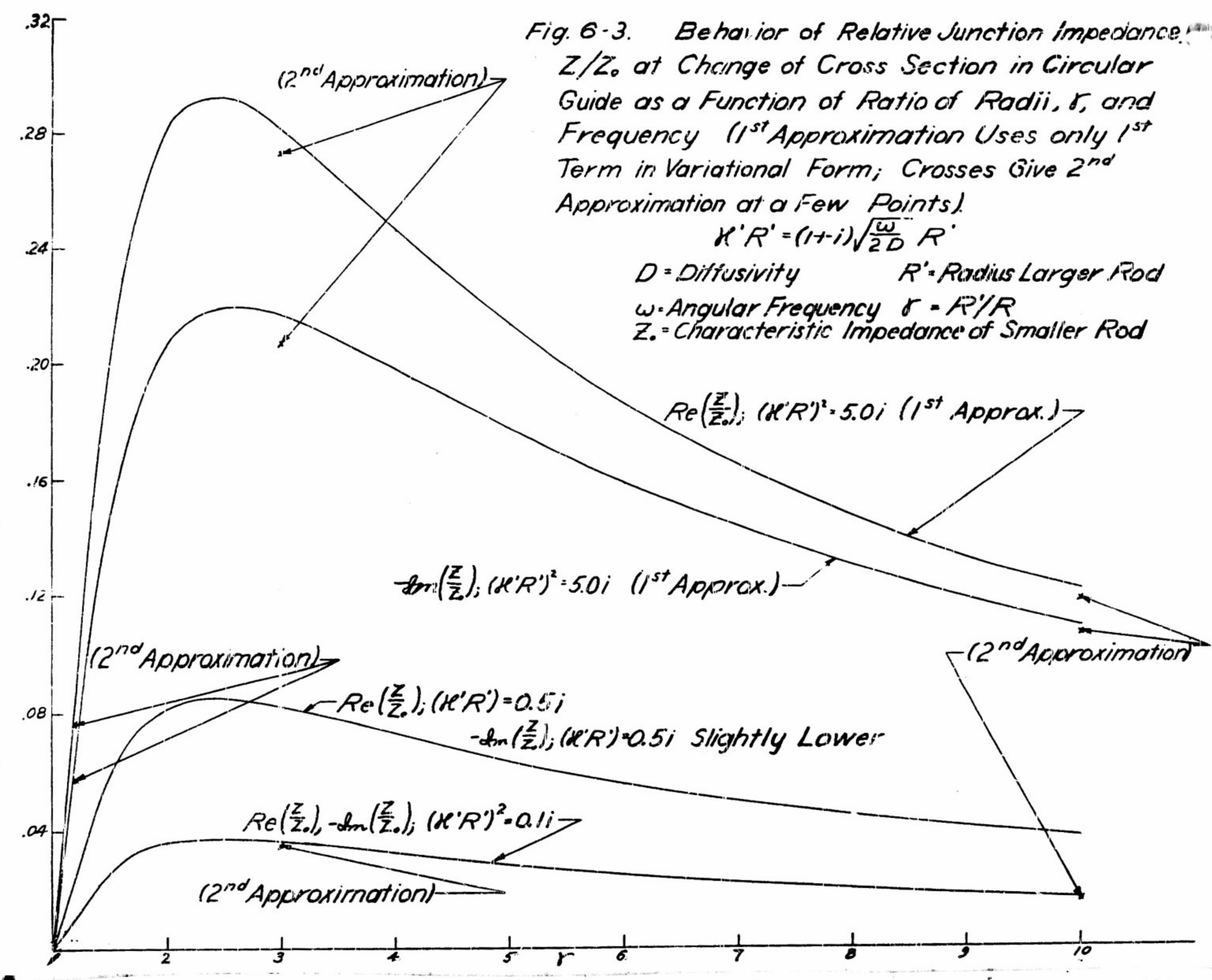


Fig. 6-3. Behavior of Relative Junction Impedance Z/Z_0 at Change of Cross Section in Circular Guide as a Function of Ratio of Radii, γ, and Frequency (1st Approximation Uses only 1st Term in Variational Form; Crosses Give 2nd Approximation at a Few Points).

$$\kappa' R' = (1+i)\sqrt{\frac{\omega}{2D}}\, R'$$

D = Diffusivity $\quad$ R' = Radius Larger Rod

ω = Angular Frequency $\quad$ $\gamma = R'/R$

Z_0 = Characteristic Impedance of Smaller Rod

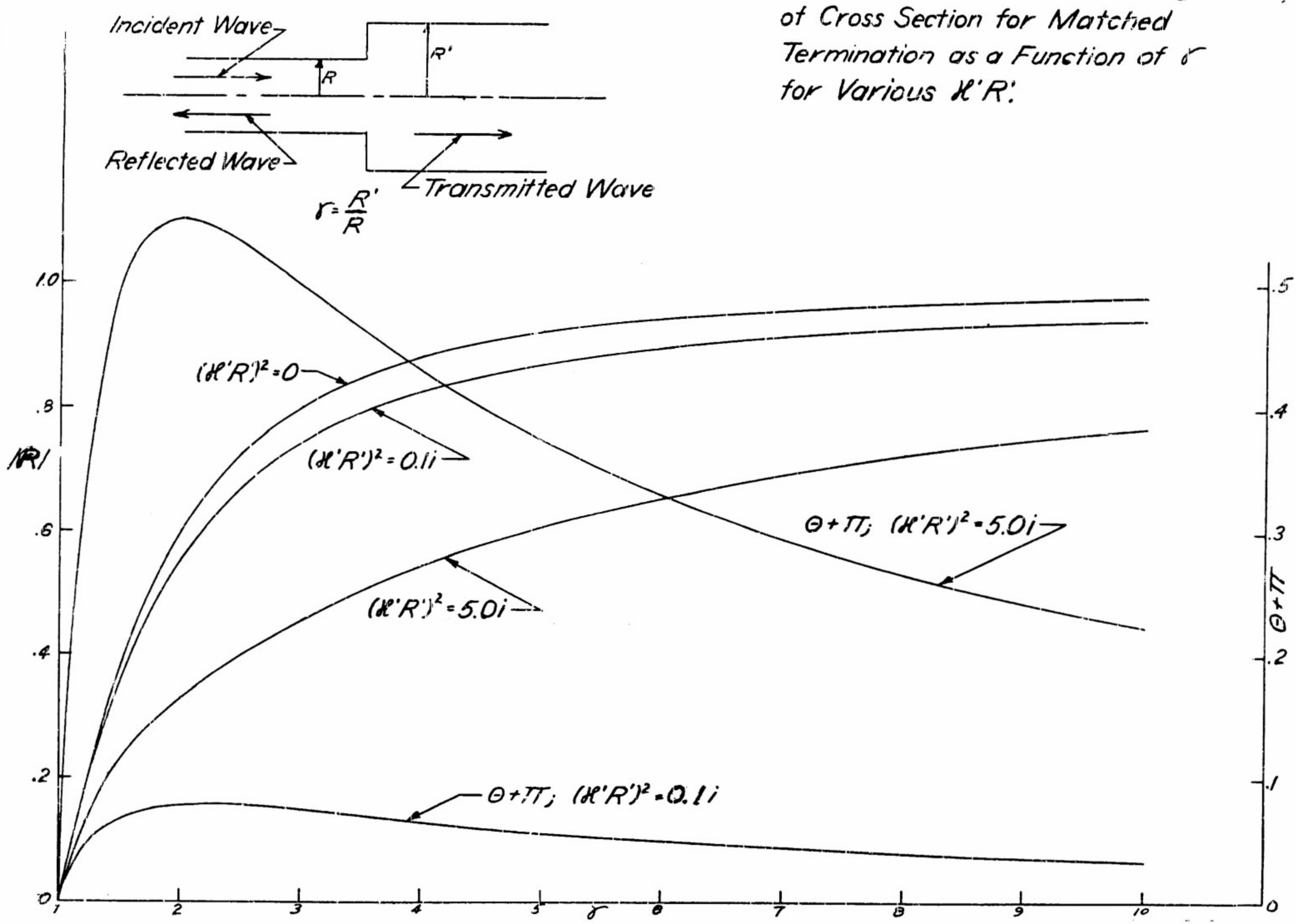


Fig. 6-4 Behavior of Reflection Coefficient $\mathcal{R} = |\mathcal{R}| e^{i\theta}$ at Change of Cross Section for Matched Termination as a Function of γ for Various $\varkappa' R'$.

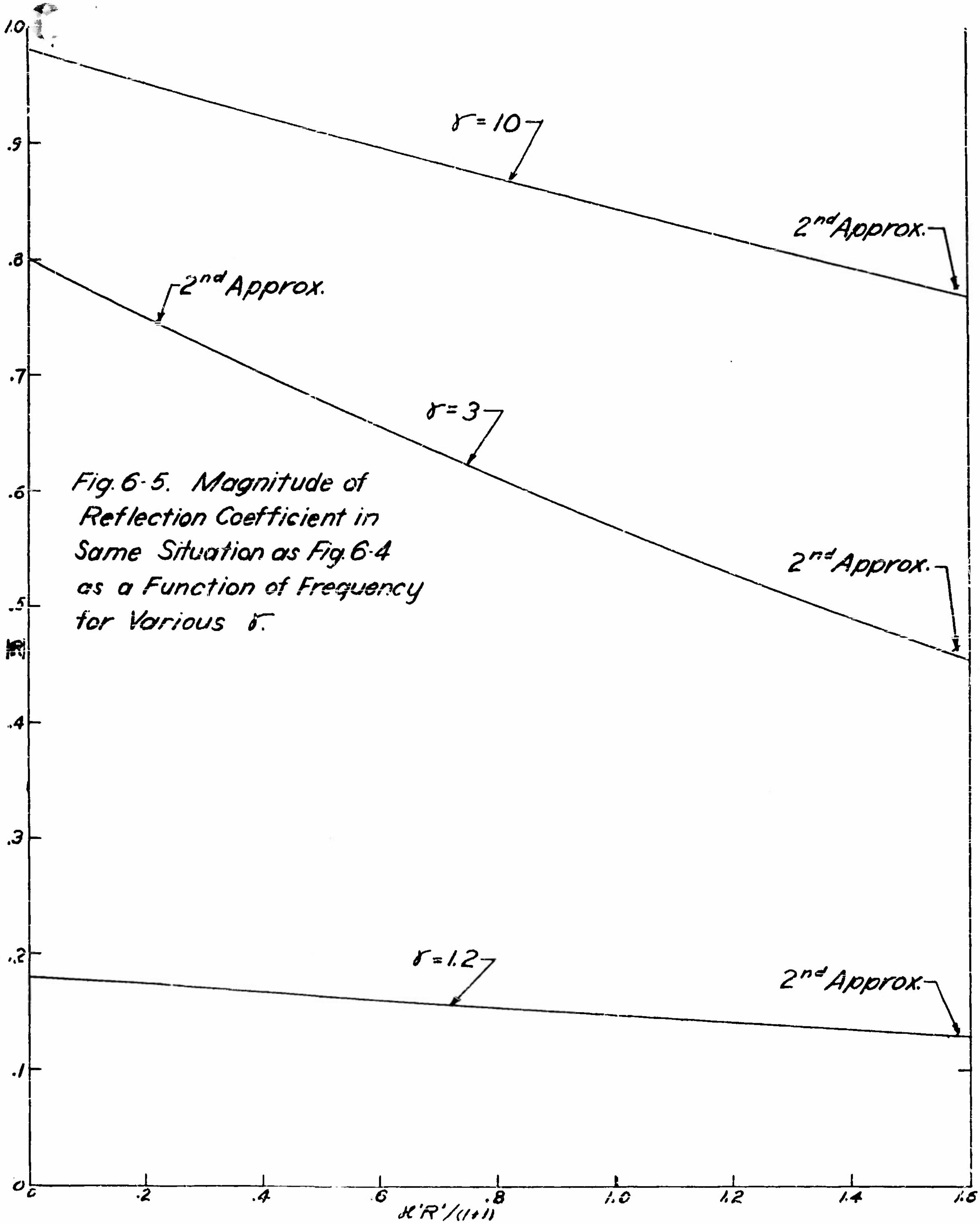


Fig. 6-5. Magnitude of Reflection Coefficient in Same Situation as Fig. 6-4 as a Function of Frequency for Various γ.

(6j) Higher Order Coefficients in the Mode Amplitude Equations

The general coefficient in equations (5.31), $C_{\ell m}$, must be evaluated to obtain higher mode amplitudes and higher order approximations to the variational expression for Z, (6.1). The general formulas from (6.2) and (6.3), are

$$C_{\ell m} = \frac{4K}{K'} R' J_0(\beta_\ell) J_0(\beta_m) \sum_{n=1}^{\infty} \frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \frac{1}{\left(1-\frac{\gamma^2\beta_\ell^2}{\beta_n^2}\right)\left(1-\frac{\gamma^2\beta_m^2}{\beta_n^2}\right)\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{1/2}}$$

$$= \frac{4K}{K'} R' J_0(\beta_\ell) J_0(\beta_m) \mathcal{S}^{(\ell,m)} \qquad (6.53)$$

$$\mathcal{S}^{(\ell,m)} = \sum_{n=1}^{\infty} \mathcal{S}_n^{(\ell,m)}; \quad \mathcal{S}_n^{(\ell,m)} = S_n^{(\ell,m)}\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{-\frac{1}{2}}, \quad S_n^{(\ell,m)} = S_n^{(0,0)} \Big/ \left(1-\frac{\gamma^2\beta_\ell^2}{\beta_n^2}\right)\left(1-\frac{\gamma^2\beta_m^2}{\beta_n^2}\right) \qquad (6.54)$$

$$S_n^{(0,0)} \equiv S_n = J_1^2(\beta_n/\gamma) \Big/ \beta_n^3 J_0^2(\beta_n) \qquad (6.55)$$

In particular

$$C_{01} = \frac{4K}{K'} R' J_0(\beta_1) \sum_{n=1}^{\infty} \frac{S_n\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{-\frac{1}{2}}}{\left(1-\frac{\gamma^2\beta_1^2}{\beta_n^2}\right)} = \frac{4K}{K'} R' J_0(\beta_1) \sum_{n=1}^{\infty} S_n^{(0,1)}\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{-\frac{1}{2}} \qquad (6.56)$$

$$C_{11} = \frac{4K}{K'} R' J_0^2(\beta_1) \sum_{n=1}^{\infty} \frac{S_n\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{-\frac{1}{2}}}{\left(1-\frac{\gamma^2\beta_1^2}{\beta_n^2}\right)^2} = \frac{4K}{K'} R' J_0^2(\beta_1) \sum_{n=1}^{\infty} S_n^{(1,1)}\left[1-\frac{\varkappa'^2 R'^2}{\beta_n^2}\right]^{-\frac{1}{2}} \qquad (6.57)$$

Note that if the new factors in the denominator (of $S_n^{(\ell,m)}$ compared to S_n) vanish or become small, the corresponding term in $C_{\ell m}$ is still not singular because the numerator vanishes or is small. Thus if

$$\beta_n = \gamma\beta_\ell + \epsilon$$

then

$$J_1(\beta_n/\gamma) = J_1(\beta_\ell + (\epsilon/\gamma)) \cong J_1(\beta_\ell) + J_1'(\beta_\ell)(\epsilon/\gamma) = \frac{\epsilon}{\gamma} J_0(\beta_\ell) \qquad (6.58)$$

and the term in $S^{(1,m)}$ is:

$$\frac{\epsilon^2}{\gamma^2} \frac{J_0^2(\beta_\ell)}{\beta_n^3 J_0^2(\beta_n)} \frac{1}{\frac{2\epsilon}{\gamma\beta_\ell}\left(1-\frac{\gamma^2\beta_m^2}{\beta_n^2}\right)} = \frac{\epsilon}{2\gamma} \frac{\beta_\ell J_0^2(\beta_\ell)}{\beta_n^3 J_0^2(\beta_n)} \frac{1}{\left(1-\frac{\gamma^2\beta_m^2}{\beta_n^2}\right)} \qquad (6.59)$$

which is small of order ϵ. The corresponding term in $S^{(\ell,\ell)}$ will not be small, however, but it will be finite.

(6k) Summing the Asymptotic Series for the Higher Order Coefficients, C_{om}.

We give now a procedure for summing the asymptotic series for the higher order coefficients. The formulas are somewhat complicated, and would be most useful when the corrections to the first term of the variational expression for Z are no longer small, hence must be accurately evaluated. This occurs when the wavelength becomes comparable to or smaller than a rod diameter and several modes are excited.

These formulas are also of interest for two other reasons. First, they provide exact sum formulas for the coefficients of the mode amplitude equations in the case of the change of section of a cylindrical rod of rectangular cross section, where an exactly analogous development is possible(the change of section must take place in only one of the rectangular dimensions). It should be remarked, however, that more powerful approaches are possible in this case, namely the solution by the equivalent static method and conformal mapping. Second, the formulas provide exact sums for the Hahn functions, which were conveniently introduced in the solution of a cavity resonator problem by mode matching techniques. These functions are defined later and the appropriate sum formulas are given.

For definiteness we take the coefficient C_{01} and write the asymptotic series in the form

$$\frac{C_{01}}{\frac{4K}{K'}R'J_0(\beta_1)} = \mathcal{S}^{(0,1)} \sim \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{(1-\sin(n+1/4)z)}{(n+1/4)^3(1-\gamma^2\beta_1^2/(n+1/4)^2\pi^2)} = \tilde{S}^{(0,1)}\ ;\ z = \frac{2\pi}{\gamma} \tag{6.60}$$

A proce e for summation of this series is the following. We have

$$\tilde{S}^{(0,1)} = \frac{\gamma}{2\pi^3}\left[\sum_{n=1}^{\infty}\frac{1}{(n+1/4)(n+1/4+\kappa)(n+1/4-\kappa)} - \sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{(n+1/4)(n+1/4+\kappa)(n+1/4-\kappa)}\right. \tag{6.61}$$

where $\kappa = (\gamma\beta_1/\pi)$ is approximately $(5\gamma/4)$ (we shall use the notation $\tilde{\kappa} = 5\gamma/4$) and $\kappa z = 2\beta_1$.

Now

$$\frac{1}{(n+1/4)(n+1/4+\kappa)(n+1/4-\kappa)} = \frac{1}{2\kappa^2}\left[\frac{1}{n+1/4+\kappa}+\frac{1}{n+1/4-\kappa}-\frac{2}{n+1/4}\right] \tag{6.62}$$

hence, using (6.14)

$$\sum_{n=1}^{\infty}\frac{1}{(n+1/4)(n+1/4+\kappa)(n+1/4-\kappa)} = \frac{1}{2\kappa^2}\left[-\Psi(1/4+\kappa)-\Psi(1/4-\kappa)+2\Psi(1/4)\right]. \tag{6.63}$$

Again using (6.61) the other term in (6.61) is

$$\sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{(n+1/4)(n+1/4+\kappa)(n+1/4-\kappa)} = \frac{1}{2\kappa^2}\sum_{n=1}^{\infty}\left[\frac{\sin(n+1/4)z}{n+1/4+\kappa}+\frac{\sin(n+1/4)z}{n+1/4-\kappa}-\frac{2\sin(n+1/4)z}{n+1/4}\right] \tag{6.64}$$

The last term in (6.64) has already been evaluated and the two expansions for it in (6.34) and (6.42) are

$$-\frac{1}{F^2}G(\tfrac{1}{4},z) = -\frac{1}{F^2}\sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{n+1/4} = -\frac{1}{F^2}\left\{\frac{\pi}{2}-4\sin\left(\frac{z}{4}\right)+\sum_{n=0}^{\infty}\frac{E_n}{(2n+1)!}\left(\frac{z}{4}\right)^{2n+1}\right\}$$

$$= -\frac{1}{F^2}\left\{\frac{\pi}{2}-4\sin\left(\frac{z}{4}\right)+\ln 2\cdot\ln\left(\frac{\pi}{2}-\frac{z}{4}\right)-\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n)!}B_n\left(\frac{\pi}{2}-\frac{z}{4}\right)^{2n}\right\} \qquad (6.65)$$

where the new function of two variables G(a,z) has been introduced.

The other two terms in (6.64) are of the form

$$\sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{n+a} = \cos\left(\tfrac{1}{4}-a\right)z\sum_{n=1}^{\infty}\frac{\sin(n+a)z}{n+a}+\sin\left(\tfrac{1}{4}-a\right)z\sum_{n=1}^{\infty}\frac{\cos(n+a)z}{n+a}$$

$$\equiv \cos\left(\tfrac{1}{4}-a\right)z\,G(a,z)+\sin\left(\tfrac{1}{4}-a\right)z\,F(a,z) \qquad (6.66)$$

where another function of two variables, F(a,z), is introduced.

Put $a = c+1/4$, $b = c-1/4$, and we have

$$\sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{n+1/4+F}+\sum_{n=1}^{\infty}\frac{\sin(n+1/4)z}{n+1/4-F} = \cos Fz\left(G(a,z)+G(-b,z)\right)-\sin Fz\left(F(a,z)-F(-b,z)\right) \qquad (6.67)$$

The sums F and G may be evaluated as follows. From (6.25)

$$\sum_{n=0}^{\infty}\cos(n+a)z \;(=)\; \frac{1}{2}\left(\cos az-\sin az\cot\frac{z}{2}\right)$$

$$\sum_{n=0}^{\infty}\sin(n+a)z \;(=)\; \frac{1}{2}\left(\cos az\cot\frac{z}{2}+\sin az\right) \qquad (6.68)$$

and upon integration from π to z (the series (6.68) do not "converge" at z=0 or 2π)

$$G(a,z)+\frac{\sin az}{a} = \sum_{n=0}^{\infty}\frac{\sin(n+a)z}{n+a} = \sin a\pi\sum_{n=0}^{\infty}\frac{(-)^n}{n+a}+\frac{1}{2}\left[\frac{\sin az-\sin a\pi}{a}\right]-\frac{1}{2}\int_{\pi}^{z}\sin az\cot\frac{z}{2}\,dz$$

$$F(a,z)+\frac{\cos az}{a} = \sum_{n=0}^{\infty}\frac{\cos(n+a)z}{n+a} = \cos a\pi\sum_{n=0}^{\infty}\frac{(-)^n}{n+a}+\frac{1}{2}\left[\frac{\cos az-\cos a\pi}{a}\right]-\frac{1}{2}\int_{\pi}^{z}\cos az\cot\frac{z}{2}\,dz \qquad (6.69)$$

The first terms in (6.69) are given by

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n+a} = \sum_{m=1}^{\infty} \frac{1}{2m+a-2} - \sum_{m=1}^{\infty} \frac{1}{2m+a-1} = -\frac{1}{2}\Psi\left(\frac{a-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{a-1}{2}\right) \qquad (6.70)$$

The integrals in(6.69) are put in a form for numericalevaluation by two different methods in Appendix II. One procedure expands $\cot z/2$ in powers of z, and integrates term by term, obtaining a rapidly convergent series over the range of interest of z. The other method sets up a numerical integration scheme specially adapted for integrands with trigonometric factors $\sin az$ and $\cos az$. The method (due to Filon) is as simple and accurate as Simpson's rule applied to the integrand without the oscillating factor.

In both cases it is advisable to subtract the singular part of $\cot z/2$ at $z=0$, namely $(2/z)$, and evaluate this term separately in terms of cosine and sine integrals. This gives

$$\frac{1}{2}\int_{\pi}^{z} e^{iaz}\cot\frac{z}{2}\,dz = Ci(az) - Ci(a\pi) + i\left(Si(az) - Si(a\pi)\right) + \int_{z}^{\pi} e^{iaz}\left(\frac{1}{z} - \frac{1}{2}\cot\frac{z}{2}\right)dz \qquad (6.71)$$

where the integrand $1/z - 1/2 \cot z/2$ is a finite, smooth, monotonic function of z over 0 to π ranging from 0 at $z=0$ to $.32$ at $z=\pi$. It is shown by $II.1$ that only values of z between 0 and π need be considered in (6.71)

The sums $G(-b,z)$ and $F(-b,z)$ are given by (6.69) with $-b$ substituted for a. Then

$$G(-b,z) = \frac{-\sin bz}{b} - \sin b\pi\left[-\frac{1}{2}\Psi\left(\frac{-b-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{-b-1}{2}\right)\right] + \frac{1}{2}\left(\frac{\sin bz - \sin b\pi}{b}\right) + \frac{1}{2}\int_{\pi}^{z}\sin bz\cot\frac{z}{2}\,dz \qquad (6.72)$$

The negative arguments in the Ψ functions can be removed by use of the relation

$$\Psi(-x) = \Psi(x-1) + \pi\cot\pi x = \Psi(x-2) + \frac{1}{x-1} + \pi\cot\pi x\,. \qquad (6.73)$$

Hence

$$-\frac{1}{2}\Psi\left(\frac{-b-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{-b-1}{2}\right) = -\frac{1}{2}\Psi\left(\frac{b-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{b-1}{2}\right) - \frac{1}{b} - \frac{\pi}{\sin\pi b} \qquad (6.74)$$

Thus, from (6.69), (6.70)

$$G(b,z) = \sin b\pi\left(-\frac{1}{2}\Psi\left(\frac{b-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{b-1}{2}\right)\right) - \frac{1}{2}\frac{(\sin bz + \sin b\pi)}{b} - \frac{1}{2}\int_{\pi}^{z}\sin bz\cot\frac{z}{2}\,dz \qquad (6.75)$$

and using (6.74)

$$G(-b,z) = -G(b,z) + \pi - \frac{\sin bz}{b} \qquad (6.76)$$

Similarly

$$F(b,z) = \cos b\pi\left(-\frac{1}{2}\Psi\left(\frac{b-2}{2}\right) + \frac{1}{2}\Psi\left(\frac{b-1}{2}\right)\right) - \frac{1}{2}\frac{(\cos bz + \cos b\pi)}{b} - \frac{1}{2}\int_{\pi}^{z}\cos bz\cot\frac{z}{2}\,dz \qquad (6.77)$$

$$F(-b,z) = F(b,z) + \frac{\cos bz}{b} - \pi\cot b\pi\,. \qquad (6.78)$$

Collecting terms from (6.64) (6.65), (6.67), (6.75) to (6.78)

$$\sum_{n=1}^{\infty} \frac{\sin(n+1/4)z}{(n+1/4)(n+1/4+c)(n+1/4-c)} = -\frac{1}{c^2} G(\tfrac{1}{4},z) + \frac{1}{2c^2}\cos cz\left(G(a,z)-G(b,z)+\pi - \frac{\sin bz}{b}\right)$$

$$-\frac{1}{2c^2}\sin cz\left(F(a,z)-F(b,z)+\pi\cot b\pi - \frac{\cos bz}{b}\right) \qquad (6.79)$$

whereas

$$\sum_{n=1}^{\infty} \frac{1}{(n+1/4)(n+1/4+c)(n+1/4-c)} = -\frac{1}{2c^2}\left[\Psi(a)+\Psi(b)-\frac{1}{b}+\pi\cot b\pi - 2\Psi(\tfrac{1}{4})\right] \qquad (6.80)$$

and

$$\tilde{S}^{(0,1)} = \frac{\delta}{2\pi^3}\left[(6.80)-(6.79)\right]; \quad a = c+1/4, \quad b = c-1/4 \quad . \qquad (6.81)$$

(6.79) to (6.81) with (6.77), (6.79) and the procedures of Appendix II permit evaluation of the asymptotic series for C_{01}, or more generally for C_{om}, just as (6.59) provides for C_{00}.

(6ℓ) Difficulties for b Nearly Integral

We note one further complication. If by accident b should approximate an integer, cot $b\pi$ diverges. Now the two terms in (6.79) and (6.80) containing cot $b\pi$ do not cancel if sin cz is not unity, and the asymptotic series is then divergent. However, it was proved in (6.59) that the terms of the original series $S^{(0,1)}$, are always finite, so that the error comes in through the use of the asymptotic forms. Accordingly the asymptotic series should be corrected around this sensitive term by removing these terms and putting in correct ones, i. e. by adding to $\tilde{S}^{(0,1)}$ the sum

$$\sum_{n \cong b}\left(\frac{J_1^2(\beta_n l/r)}{\beta_n^3 J_0^2(\beta_n)(1-\frac{c^2\beta_n^2}{\beta_n^2})} - \frac{\delta}{2\pi^3}\frac{(1-\sin(n+1/4)z)}{(n+1/4)(n+1/4+c)(n+1/4-c)}\right) . \qquad (6.82)$$

As many terms of (6.82) should be added, for n above and below b, as contribute in the decimal place of interest. Note that if b = c-1/4 = ℓ - ε, the term n=ℓ in $\tilde{S}^{(0,1)}$ is approximately $\delta/2\pi^3\,(1-\sin cz)/2c^2\epsilon$ which on subtraction from the sum formula cancels the sum of the two cot $b\pi$ terms in (6.79), (6.80) Thus any singularity is removed.

If we make the further approximation in $\tilde{S}^{(0,1)}$ of using the asymptotic form of c, namely $\tilde{c} = 5\delta/4$, then $\tilde{c}z = 5\pi/2$, $\cos\tilde{c}z = 0$, $\sin\tilde{c}z = 1$, and (6.79) to (6.81) give

$$\tilde{S}^{(0,1)} = \frac{\delta}{2\pi^3}\sum_{n=1}^{\infty}\frac{1-\sin(n+1/4)z}{(n+1/4)(n+1/4+5\delta/4)(n+1/4-5\delta/4)}$$

$$= \frac{\delta}{4\pi^3\tilde{c}^2}\left[-\Psi(\tilde{a})-\Psi(\tilde{b})+2\Psi(\tfrac{1}{4})+\frac{1}{\tilde{b}}+F(\tilde{a},z)-F(\tilde{b},z)-\frac{\cos\tilde{b}z}{\tilde{b}}+2G(\tfrac{1}{4},z)\right] \qquad (6.83)$$

where $\tilde{\mu} = 5\gamma/4$, $a = \tilde{\mu} + 1/4$, $b = \tilde{\mu} - 1/4$, $z = 2\pi/\gamma$. In (6.83) the possible

singularities at b = an integer have cancelled out, but at the cost of a further approximation which may be a non-negligible fraction of each term of the series (actually a few percent since $c = 1.220\gamma$ vs. $\tilde{\mu} = 1.250\gamma$). The fraction decreases with increasing n, however, once the critical term ($n \cong b$) has been passed, so that for small γ (less than three or four say), (6.83) might be quite usable as an asymptotic series for $S^{(0,1)}$ or $\mathcal{S}^{(0,1)}$. A numerical example applying (6.83) to calculation of $\tilde{S}^{(0,1)}$ is given in Appendix III.

(6m) Evaluation of $C_{\ell\ell}$

Application of this above procedure to evaluation of more general $C_{\ell m}$ will prove very simple when $l \neq m$, but more difficult for $C_{\ell\ell}$. Consider the latter first, and in particular the prototype case C_{11} . From (6.57) the asymptotic series for C_{11} is

$$\frac{C_{11}}{4\frac{K}{K'} R' U_0^2(\beta_1)} \sim \tilde{S}^{(1,1)} = \frac{\gamma}{2\pi^3} \sum_{n=1}^{\infty} \frac{(1-\sin(n+1/4)z)(n+1/4)}{(n+1/4+\mu)^2 (n+1/4-\mu)^2} \tag{6.84}$$

$$= \gamma/2\pi^3 \sum_{n=1}^{\infty} \frac{(1-\sin(n+1/4)z)}{4\mu} \left[\frac{1}{(n+1/4-\mu)^2} - \frac{1}{(n+1/4+\mu)^2}\right] \tag{6.85}$$

$$= \frac{\gamma}{8\pi^3\mu}\left[\Psi'(-b) - \Psi'(a) - \cos\mu z\left(G_2(-b,z) - G_2(a,z)\right) - \sin\mu z\left(F_2(-b,z) + F_2(a,z)\right)\right] \tag{6.86}$$

where

$$\Psi'(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2} \tag{6.87}$$

$$\Psi'(-b) = \sum_{n=1}^{\infty} \frac{1}{(n-b)^2} = -\Psi'(b-1) + \frac{\pi^2}{\sin^2 b\pi} = -\Psi'(b) - \frac{1}{b^2} + \frac{\pi^2}{\sin^2 b\pi} \tag{6.88}$$

and

$$G_2(a,z) = \sum_{n=1}^{\infty} \frac{\sin(n+a)z}{(n+a)^2} \; ; \quad F_2(a,z) = \sum_{n=1}^{\infty} \frac{\cos(n+a)z}{(n+a)^2} . \tag{6.89}$$

Integrating (6.77) from 0 to z yields

$$G_2(a,z) = z\cos a\pi\left[-\tfrac{1}{2}\Psi\left(\tfrac{a-?}{2}\right) + \tfrac{1}{2}\Psi\left(\tfrac{a-1}{2}\right) - \tfrac{1}{2a}\right] - \frac{1}{2}\frac{\sin az}{a^2}$$

$$-\frac{1}{2}z\int_{\pi}^{z} \cos az \cot z/2 \, dz + \frac{1}{2}\int_{0}^{z} \cos az \, z \cot z/2 \, dz \tag{6.90}$$

The first integral in (6.90) has been given in (II .10) By a similar procedure we obtain

$$\frac{1}{2}\int_0^z \cos az\, z \cot z/2\, dz = \frac{\sin az}{a} + \sum_{n=1}^{\infty} \frac{(-)^n B_n}{a^{2n+1}}\left(\sin az\, \overline{\mathrm{Cos}}_{n+1}(az) - \cos az\, \overline{\mathrm{Sin}}_{n+1}(az)\right) \qquad (6.91)$$

A numerical integration procedure can also be set up.

Integrating (6.75) from 0 to z gives

$$F_2(az) = \Psi'(a) - z \sin a\pi\left[-\tfrac{1}{2}\Psi\left(\tfrac{a-2}{2}\right) + \tfrac{1}{2}\Psi\left(\tfrac{a-1}{2}\right) - \tfrac{1}{2a}\right] - \frac{\cos az - 1}{2a^2}$$

$$+ \frac{z}{2}\int_\pi^z \sin az \cot z/2\, dz - \frac{1}{2}\int_0^z \sin az\, z \cot z/2\, dz \qquad (6.92)$$

where the first integral is given by (II.11) and the second integral by

$$\frac{1}{2}\int_0^z \sin az\, z \cot z/2\, dz = \frac{1-\cos az}{a} - \sum_{n=1}^{\infty} \frac{(-)^n B_n}{a^{2n+1}}\left(\cos az\, \overline{\mathrm{Cos}}_{n+1}(az) + \sin az\, \overline{\mathrm{Sin}}_n(az)\right) \qquad (6.93)$$

The formulas for $G_2(-b,z)$, $F_2(-b,z)$ are then obtained by integrating (6.78), (6.76) respectively. This gives immediately

$$G_2(-b,z) = G_2(b,z) + \frac{\sin bz}{b^2} - \pi z \cot b\pi \qquad (6.94)$$

and, using (6.88)

$$F_2(-b,z) = -F_2(b,z) - \pi z - \frac{\cos bz}{b^2} + \frac{\pi^2}{\sin^2 b\pi} \qquad (6.95)$$

Finally putting (6.88) (6.94) and (6.95) into (6.86) we have

$$\xi^{(1,1)} = \frac{\gamma}{8\pi^3\kappa}\Big[-\Psi'(b) - \Psi'(0) - \frac{1}{b^2} + \frac{\pi^2}{\sin^2 b\pi} - \cos\kappa z\left(G_2(b,z) - G_2(a,z) + \frac{\sin bz}{b^2} - \pi z \cot b\pi\right)$$

$$-\sin\kappa z\left(F_2(b,z) + F_2(a,z) - 2\Psi'(b) - \pi z - \frac{\cos bz}{b^2} + \frac{\pi^2}{\sin^2 b\pi}\right) \qquad (6.96)$$

Note that in (6.96) for the special case $\kappa z = 5\pi/2$, the singularity when b is an integer cancels out as in (6.83). However, in the case of general κz, we have the singularity

$$\frac{\gamma}{8\pi^3\kappa}\left[\frac{\pi^2}{\sin^2 b\pi}(1 - \sin\kappa z) + \cos\kappa z\,(\pi z \cot b\pi)\right] \qquad (6.97)$$

If the term $\frac{\gamma}{2\pi^3}\frac{(1-\sin(\ell+1/4)\zeta)}{4\kappa}\cdot\frac{1}{(\ell+1/4-\kappa)^2}$ in (6.85), for which $b=\kappa-1/4=\ell-\epsilon$ is subtracted from the sum, this subtracts the singular terms $\frac{\gamma}{8\pi\kappa\ell}\left[\frac{1-\sin\kappa\zeta}{\epsilon^2}-\frac{\zeta\cos\kappa\zeta}{\epsilon}\right]$

from the same term in (6.97), and removes the singularity. As in (6.82), the correct Bessel function terms can be used in place of the asymptotic form, for the terms near ℓ.

(6n) Evaluation of $C_{\ell m}$ $\ell\neq m$

The calculation of $C_{\ell m}$ can be made to depend in a direct, simple way on $C_{o\ell}$ and C_{om}, so that it is unnecessary to sum the asymptotic series. We have, on transforming (6.53) by partial fractions,

$$C_{\ell m}=\frac{4K}{K'}R'J_0(\beta_\ell)J_0(\beta_m)\sum_{n=1}^{\infty}\frac{J_1^2(\beta_n/\gamma)\left[1-\frac{\kappa'^2R'^2}{\beta_n^2}\right]^{-1/2}}{J_0^2(\beta_n)\,\beta_n(\beta_\ell^2-\beta_m^2)}\left[\frac{\beta_\ell^2}{\beta_n^2-\gamma^2\beta_\ell^2}-\frac{\beta_m^2}{\beta_n^2-\gamma^2\beta_m^2}\right]$$

$$=\frac{1}{\beta_\ell^2-\beta_m^2}\left[\beta_\ell^2J_0(\beta_m)C_{o\ell}-\beta_m^2J_0(\beta_\ell)C_{om}\right];\quad \ell\neq m \tag{6.98}$$

Thus the calculation of the two dimensional array of coefficients $C_{\ell m}$ in the linear equations (5.31), is reduced essentially to the calculation of two linear sequences of coefficients $C_{o\ell}$, and $C_{\ell\ell}$.

(6o) The Hahn Functions

We note that the procedure introduced in the last sections to sum the Fourier series asymptotic to $C_{\ell m}$ can also be applied to sum the Hahn functions introduced some years ago by W. Hahn in the solution of a cavity resonator problem by mode expansions and field matching methods.[11]

Hahn's functions are defined as

$$S_p(a)=\sum_{m=1}^{\infty}\frac{\sin^2\pi ma}{m\left(\frac{a^2m^2}{p^2}-1\right)}\;;\quad U_p(a)=\sum_{m=1}^{\infty}\frac{a^2m\sin^2\pi ma}{p^2\left(\frac{a^2m^2}{p^2}-1\right)^2};\; p \text{ an integer} \tag{6.99}$$

and in the reference a sum formula for $S_p(a)$ is given but not one for $U_p(a)$.

By analysis precisely like (6.61) - (6.64) we have

$$S_p(a)=\frac{1}{4}\sum_{m=1}^{\infty}(1-\cos m\zeta)\left[\frac{1}{m+\sigma}+\frac{1}{m-\sigma}-\frac{2}{m}\right];\quad \begin{cases}\zeta=2\pi a\\ \sigma=p/a\end{cases} \tag{6.100}$$

and using (6.14), (6.73). we have

$$\frac{1}{4}\sum_{m=1}^{\infty}\left(\frac{1}{m+\sigma}+\frac{1}{m-\sigma}-\frac{2}{m}\right)=\frac{1}{4}\left[-2\Psi(\sigma)+\frac{1}{\sigma}-\pi\cot\pi\sigma+{}^{-}\Psi(0)\right] \tag{6.101}$$

(11)W. Hahn - A New Method for the Calculation of Cavity Resonators - J. App. Physics, 12 62 (1941).

while analysis like (6.66), (6.67), with (6.76) and (6.78) gives

$$\frac{1}{4}\sum_{m=1}^{\infty}\cos m\zeta\left[\frac{1}{m+\sigma}+\frac{1}{m-\sigma}-\frac{2}{m}\right]=\frac{1}{4}\Big[\cos\sigma\zeta\left(2F(\sigma,\zeta)+\frac{\cos\sigma\zeta}{\sigma}-\pi\cot\sigma\pi\right)$$

$$+\sin\sigma\zeta\left(2G(\sigma,\zeta)-\pi+\frac{\sin\sigma\zeta}{\sigma}\right)+\ln 2(1-\cos\zeta)\Big]. \qquad (6.102)$$

(6.101) and (6.102) in (6.100) give the sum for general values of σ and ζ. However, in the original definition of $S_p(a)$ p is taken to be an integer, which leads to a simplification. Thus

$$\sigma\zeta = 2\pi p, \quad \cos\sigma\zeta = 1, \quad \sin\sigma\zeta = 0 \qquad (6.103)$$

and

$$S_p(a) = -\frac{1}{2}\Psi(\sigma)-\frac{1}{2}F(\sigma,\zeta)-\frac{1}{4}\ln 2(1-\cos\zeta)-\frac{C_E}{2}$$

$$\equiv -\frac{1}{2}\Psi(p/a)-\frac{1}{2}F(p/a,2\pi a)-\frac{1}{4}\ln[2(1-\cos 2\pi a)]-\frac{C_E}{2} \quad (p \text{ an integer}) \qquad (6.104)$$

where $F(p/a, 2\pi a)$ is evaluated by (6.77), and (II.10), and $\Psi(0) = -C_E = -.577216$ has been used

We note the special case a=1/2 for which

$$S_p(1/2) = -1/2\,\Psi(2p)-1/2\,F(2p,\pi)-\frac{1}{4}\ln 4-\frac{C_E}{2},$$

Now

$$F(2p,\pi) = \left[-\frac{1}{2}\Psi(p-1)+\frac{1}{2}\Psi(p-1/2)\right]-\frac{1}{2p}$$

hence

$$S_p(1/2) = -\frac{1}{2}\Psi(p-1/2 - \ln 2 - \frac{C_E}{2}; \quad p \text{ an integer} \qquad (6.105)$$

a result which is readily checked directly from (6.99).

Similarly, we have , using partial fractions just as in (6.85)

$$U_p(a) = \frac{\sigma}{8}\sum_{m=1}^{\infty}(1-\cos m\zeta)\left[\frac{1}{(m-\sigma)^2}-\frac{1}{(m+\sigma)^2}\right]; \quad \sigma = p/a, \ \zeta = 2\pi a \qquad (6.106)$$

which becomes, using (6.87), and a formula like (6.67)

$$= \frac{\sigma}{8}\left[\Psi'(-\sigma) - \Psi'(\sigma) - \cos\sigma z\left(F_2(-\sigma,z) - F_2(\sigma,z)\right) + \sin\sigma z\left(G_2(-\sigma,z) + G_2(\sigma,z)\right)\right]$$

and using (6.88), (6.95), and (6.94) to remove the negative arguments, gives

$$U_p(a) = \frac{\sigma}{8}\left[-2\Psi'(\sigma) - \frac{1}{\sigma^2} + \frac{\pi^2}{\sin^2\sigma\pi} + \cos\sigma z\left(2F_2(\sigma,z) + \pi z + \frac{\cos\sigma z}{\sigma^2} - \frac{\pi^2}{\sin^2\sigma\pi}\right)\right.$$

$$\left. + \sin\sigma z\left(2G_2(\sigma,z) + \frac{\sin\sigma z}{\sigma^2} - \pi z\cot\sigma\pi\right)\right] \tag{6.107}$$

(6.107) holds for arbitrary σ and z, as in the case of (6.102), but for the special values (6.103), it simplifies to

$$U_p(a) \equiv \frac{\pi^2 p}{4} + \frac{p}{4a}\left(F_2(p/a, 2\pi a) - \Psi'(p/a)\right); \qquad p \text{ an integer} \tag{6.108}$$

provided that $\sigma = p/a$ is not an integer. When p/a is an integer, the terms containing $\frac{1-\cos\sigma z}{\sin^2\sigma\pi}$ and $\sin\sigma z\cot\sigma\pi$ make additional finite contributions, to give

$$U_p(a) = \frac{\pi^2 p}{4} + \frac{p}{4a}\left[F_2(p/a, 2\pi a) - \Psi'(p/a)\right] - \frac{\pi^2 a p}{4}; \qquad p,\ p/a \text{ integral.} \tag{6.109}$$

(6p) The Factors $\left[1 - (\kappa' R'/\beta_n)^2\right]^{-1/2}$

The calculation of Z requires summation of the various series $\mathcal{S}^{(\ell,m)}$ in which the terms S_n are multiplied by the factor involving the wavelength, $\left[1-(\kappa' R'/\beta_n)^2\right]^{-1/2}$. For long wavelengths (compared to the diameter) $\kappa' R' \ll 1 \ll \beta_n$, and the factor quickly approaches its asymptotic value of 1. But for short wavelengths, this approach may be too slow for practicable summation, and it will be useful to subtract the next term in the approach to 1, namely $\frac{1}{2}(\kappa' R'/\beta_n)^2$, and to sum the corresponding series. These series, which arise from $\mathcal{S}$ and $\mathcal{S}^{(\ell,m)}$ multiplied by this additional factor, can readily be summed since we merely require summation of such series as

$$\sum_{n=1}^{\infty} \frac{(1-\sin(n+1/4)z)}{(n+1/4)^5}, \quad \sum_{n=1}^{\infty} \frac{(1-\sin(n+1/4)z)}{(n+1/4)^3\left[(n+1/4)^2 - \mu^2\right]}, \quad \sum_{n=1}^{\infty} \frac{(1-\sin(n+1/4)z)}{(n+1/4)\left[(n+1/4)^2-\mu^2\right]^2}$$

(see (6.10), (6.12), (6.61), (6.84))

The first series is found by further integration of the expansions for

$$\sum_{n=1}^{\infty} \frac{\sin(n+1/4)z}{(n+1/4)z} .$$

The second requires use of the partial fraction expansion

$$\frac{1}{(n+1/4)^3[(n+1/4)^2-F^2]} = \frac{1}{2F^4}\left(\frac{1}{n+1/4+F}+\frac{1}{n+1/4-F}-\frac{2}{n+1/4}-\frac{2F^2}{(n+1/4)^3}\right)$$

and all the terms in this summation have been done previously. The third requires

$$\frac{1}{(n+1/4)(n+1/4+F)^2(n+1/4-F)^2} = \frac{1}{2F^4}\left(\frac{2}{n+1/4}-\frac{1}{n+1/4+F}-\frac{1}{n+1/4-F}\right)+\frac{1}{4F^3}\left(\frac{1}{(n+1/4-F)^2}-\frac{1}{(n+1/4+F)^2}\right)$$

all terms of which have been done previously. Finally, the sum of the series with the new factor, $\frac{1}{2}(\kappa' R'/\beta_n)^2$, multiplied into the terms of $\tilde{S}^{(\ell,m)}$, $\ell \neq m$ follows immediately from the sums of the corresponding series obtained from $\tilde{S}^{(o,\ell)}$ and $\tilde{S}^{(o,m)}$ by multiplication by the new factor, just as $\tilde{S}^{(\ell,m)}$ follows from $\tilde{S}^{(o,m)}$ and $\tilde{S}^{(o,\ell)}$ (as in (6.98)).

Then we have $\mathcal{S}^{(\ell,m)} \simeq \sum_{n=1}^{\infty} S_n^{(\ell,m)}\left(1+\frac{1}{2}\left(\frac{\kappa' R'}{\beta_n}\right)^2\right)$ where the last series is evaluated as above. The difference series (between left and right) will now converge more rapidly and permit easy numerical summation for larger values of ($\kappa' R'$).

(6q) Supplementary Note on Evaluation of $F(\sigma,z)$, $G(\sigma,z)$ and the Hahn Functions

After completion of the manuscript the author's attention was directed by J. R. Whinnery to the work of Strachey and Wallis on the Hahn functions *. By methods similar to those above, using the Ψ function in summation of series, these authors evaluate the Hahn functions. Closed expressions are obtained when "σ" is rational, a situation not considered here. For general σ and ρ similar but not identical formulas to those above are obtained, which also involve a definite integral. These may be shown equivalent to those given here by using different formulas for $F(\sigma,z)$, $G(\sigma,z)$, $F_2(\sigma,z)$, $G_2(\sigma,z)$ in (6.102) , (6.104),

Thus in place of (6.77) , and (6.75) , we have

$$F(\sigma,z) = -\ln(2\sin z/2) - C_E + \frac{1}{2\sigma} - \Psi(\sigma) - \frac{\cos\sigma z}{2\sigma} + \frac{1}{2}\int_0^z (1-\cos\sigma z)\cot z/2 \, dz$$

$$G(\sigma,z) = \frac{\pi}{2} - \frac{\sin\sigma z}{2\sigma} - \frac{1}{2}\int_0^z \sin\sigma z \cot z/2 \, dz \qquad (6.110)$$

* C. Strachey and P. J. Wallis - Hahn's Functions $S_\rho(\sigma)$ and $U_\rho(\sigma)$ - Phil. Mag. 37, 87 (1946).

obtained respectively from (6.77) and (6.75) by using the definite integrals

$$\frac{1}{2}\int_0^{\pi}(1-\cos\sigma z)\cot\frac{z}{2}\,dz = C_E + \ln 2 + \Psi(\sigma) - \frac{(1+\cos\sigma\pi)}{2\sigma} + \cos\sigma\pi\left[-\frac{1}{2}\Psi\left(\frac{\sigma-2}{2}\right)+\frac{1}{2}\Psi\left(\frac{\sigma-1}{2}\right)\right] \tag{6.111}$$

$$\frac{1}{2}\int_0^{\pi}\sin\sigma z\cot\frac{z}{2}\,dz = \frac{\pi}{2} + \frac{\sin\sigma\pi}{2\sigma} - \sin\sigma\pi\left[-\frac{1}{2}\Psi\left(\frac{\sigma-2}{2}\right)+\frac{1}{2}\Psi\left(\frac{\sigma-1}{2}\right)\right]$$

[Note: There is an error in the formula of the reference for the series

$$\sum_{n=1}^{\infty}\frac{\cos nz}{n+\sigma} \equiv \cos\sigma z\, F(\sigma,z) + \sin\sigma z\, G(\sigma,z)$$

where $F(\sigma,z)$ and $G(\sigma,z)$ are replaced by (6.110) above. The term $\frac{-\sin\sigma z}{2\sigma}$ in $G(\sigma,z)$ has been omitted.]

Putting (6.110) in (6.102) gives the general new form for $S_p(a)$, which agrees with Strachey and Wallis,

$$S_p(a) = -\sin^2\pi p\left[\Psi\left(\frac{p}{a}\right) - \frac{a}{2p} + \frac{\pi}{2}\cot\frac{p\pi}{a} + C_E + \ln(2\sin\pi a)\right] + a\int_0^{\pi}\sin p(2\pi-\phi)\sin p\phi\cot a\phi\, d\phi\, . \tag{6.112}$$

For the special case (6.103), (6.112) becomes

$$S_p(a) = -a\int_0^{\pi}\sin^2 pz\cot az\, dz = -\frac{1}{4}\int_0^{2\pi a}\left(1-\cos\frac{pz}{a}\right)\cot\frac{z}{2}\,dz\ ; \qquad p = \text{an integer} \tag{6.113}$$

which also follows directly from (6.104) on using (6.110).

Integration of (6.110) as in (6.90) and (6.92) gives

$$F_2(\sigma,z) = \Psi'(\sigma) - \frac{\pi z}{2} + \frac{1-\cos\sigma z}{2\sigma^2} + \frac{z}{2}\int_0^z \sin\sigma z\cot\frac{z}{2}\,dz - \frac{1}{2}\int_0^z z\sin\sigma z\cot\frac{z}{2}\,dz$$

$$G_2(\sigma,z) = -z\left(\Psi(\sigma) - \frac{1}{2\sigma} + C_E + \ln 2\right) - \frac{\sin\sigma z}{2\sigma^2} - z\ln\sin\frac{z}{2} \tag{6.114}$$

$$+\frac{z}{2}\int_0^z(1-\cos\sigma z)\cot\frac{z}{2}\,dz + \frac{1}{2}\int_0^z z\cos\sigma z\cot\frac{z}{2}\,dz\ .$$

Putting (6.114) in (6.107) gives a general formula for $U_p(a)$ which agrees with that of Strachey and Wallis, namely

$$U_p(a) = \frac{-\pi p}{2}\sin 2\pi p\left[\ln 2\sin\pi a + \frac{\pi}{2}\cot\frac{p\pi}{a} + \Psi(p/a) + C_E - \frac{a}{2p}\right]$$

$$-\frac{p\sin^2\pi p}{4a}\left[2\Psi'\left(\frac{p}{a}\right) + \frac{a^2}{p^2} - \frac{\pi^2}{\sin^2\frac{\pi p}{a}}\right] + \frac{ap}{2}\int_0^{\pi}\left[(\pi-\phi)\sin 2p(\phi-\pi) - \pi\sin 2\pi p\right]\cot a\phi\, d\phi \tag{6.115}$$

For p integral, but p/a non-integral, (6.115) yields the special formula

$$U_p(a) = \frac{ap}{2}\int_0^{\pi}(\pi-\phi)\sin 2p\phi \cot a\phi \, d\phi ; \qquad p \text{ an integer} \quad (6.116)$$

which is equivalent to (6.108). When both p/a and p are integers, finite contributions come from the terms in (6.115) containing $\left(\sin 2\pi p \cot \frac{p\pi}{a}\right)$ and $\left(\sin^2 \pi p / \sin^2(\pi p/a)\right)$, to give

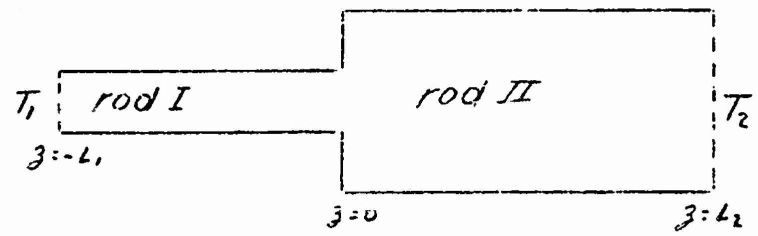
$$U_p(a) = \frac{ap}{2}\int_0^{\pi}(\pi-\phi)\sin 2p\phi \cot a\phi \, d\phi - \frac{\pi^2 ap}{4} ; \qquad p \text{ and } p/a \text{ integral} \quad (6.117)$$

equivalent to (6.109).

7. THE STATIC PROBLEM OF THE CHANGE OF CROSS SECTION

(7a) Formulation and Solution of the Problem

As the frequency approaches zero, or the wave length gets long, the solution to the temperature wave reflection problem at the junction of two circular rods goes over into the static problem of steady heat flow past such a junction produced by a steady temperature difference. In fact, the latter problem can be exactly solved by use of the equivalent circuit for the junction, although the picture of incident and reflected waves on a transmission line has broken down.



Consider the static problem in the figure, where the cross sections at $z=-L_1, L_2$ are sufficiently far from the junction plane at $z=0$, that the higher static modes, of the form $J_0(k_n r)e^{-k_n|z|}$ will be completely damped out on these cross sections. The temperature at $z=-L_1, L_2$ and away from the junction, thus takes the form $T=Az+B$ in I and $T'=A'z+B'$ in II, which is the form of the first or principal static mode, and satisfies Laplace's equation. Then we must have

$$
\begin{aligned}
T_1 &= -AL_1 + B \\
T_2 &= +A'L_2 + B'
\end{aligned}
\qquad (7.1)
$$

At $z=0$, the principal static mode takes the form $T(0)=B,\ T'(0)=B'$, while $\partial T/\partial z = A$, $\partial T'/\partial z = A'$ holds for all z including z=0. Although this static mode does not satisfy transmission line equations, we may consider that T and $\partial T/\partial z$ define a voltage and a current at the junction plane by means of the equations

$$V_0 = T(0),\quad I(0) = \frac{1}{i\mathcal{H}Z_0}\frac{\partial V}{\partial z} = -K\pi R^2\left(\frac{\partial T}{\partial z}\right)_0 \qquad (7.2)$$

on using $Z_0 = i/K\pi R^2\mathcal{H}$, so that $\mathcal{H}Z_0$ is independent of frequency. Similarly $V_0' = T'(0)$, $I'(0) = -K'\pi R'^2(\partial T'/\partial z)_0$. Now the four quantities V_0, V_0', I_0, I_0' will be related by means of the impedance matrix used in the general treatment of the junction, and in fact the equivalent circuit may be employed, but with the static forms of the circuit elements, i. e., their limits as $\omega \to 0$ when we shall put $Z \to Z_s$.

The circuit gives $I_0 = I_0'$ since it consists only of a series element or

$$K\pi R^2 A = K'\pi R'^2 A' \qquad (7.3)$$

and as in the wave problems

$$V_0 - V_0' = Z_s I_0 \quad \text{or} \quad B - B' = -K\pi R^2 A Z_s \qquad (7.4)$$

Thus we have four equations, which determine A, B, A', B' in terms of the boundary condition and the junction impedance Z_s, hence solve the static problem.

$$\left.\begin{aligned} -AL_1 + B &= T_1 \\ A'L_2 + B' &= T_2 \\ KR^2A &= K'R'^2A' \\ B - B' &= -K\pi R^2 A Z_s \end{aligned}\right\} \quad \begin{aligned} KR^2A &= \frac{(T_2 - T_1)K'R'^2KR^2}{L_2KR^2 + L_1K'R'^2 + \pi KR^2K'R'^2 Z_s} = K'R'^2A' \\ B &= T_1 + AL_1 \\ B' &= T_2 - A'L_2 \end{aligned} \tag{7.5}$$

The temperature varies linearly with z from T_1 at $z=-L_1$, until the higher modes around the junction have significant amplitudes; the principal mode part suffers a discontinuity at the junction, then varies linearly to T_2 at $z=L_2$. The heat flow is given by $-\pi KR^2A$ which is constant over the entire length of the rod. Since this remains the same even near the junction, it is evident that the higher modes make no net contribution to the heat flow. This result follows just as in the dynamic problem immediately from the relation

$$\int_0^R J_0(\gamma_n r)2\pi r\,dr = 0 \tag{7.6}$$

where the integral is proportional to the heat flow through a cross section due to the n^{th} higher mode. If in (7.5) we take $K=K'$, $R=R'$, $Z_s=0$ the heat flow becomes simply $-\pi KR^2(T_2-T_1)/(L_1+L_2)$, the result for a uniform rod.

(7b) Similarity of the Static and Dynamic Fields in the Junction Plane.

The basic reason why it has been possible to analyze the static problem in the same terms as the dynamic problem, is that the situation in the junction plane is the same for both. The modes of both problems have the same radial structure, and for a matching problem at fixed z (i.e., a junction region with no extension in the z direction) it does not matter what the z dependence is. Thus the values of T and $(\partial T/\partial z)$ in I and II arising at z=0 from the principal static mode might be thought of as coming from the principal modes of some dynamic problem at finite frequency (which have $\cos kz$, $\sin kz$ dependence), and determine the same higher mode structure in the junction plane, characterized by Z_s.

One could, in fact, repeat the analysis of Section 5 exactly for the static problem, using only different forms for the principal mode functions, and arrive at the same formulation. This is namely that the principal mode part of $-K\pi R^2(\partial T/\partial z)$ is continuous across the junction, and could be identified as the current (it is the total heat flux), and that the principal mode part of T is discontinuous across the junction, and on each side could be identified as the voltage. This discontinuity in V_0 is proportional to the continuous value of I_0 because of the linearity of the problem, and the proportionality constant represents essentially the contributions of the higher modes to the temperature field at the junction for a particular value of the heat flow, as compared to unit heat flow. (The total value of T is, of course, continuous at z=0)

The higher mode amplitudes and the complete temperature field of the static problem could be found by solving the linear equations for the A_n by some successive approximation method.

(7c) The Static Problem with a Free End.

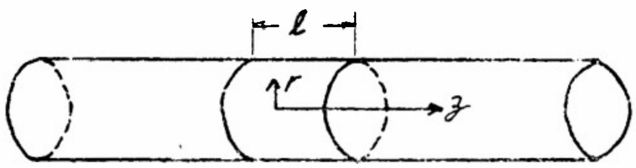
We note that if one end of the rod should be free, instead of being at a fixed temperature, then $A' = \partial T'/\partial z = 0$. Hence by (7.3) $A = 0$ as well, and by (7.4) $B = B' = T_1$. Thus the temperature is constant throughout the rods, regardless of the presence of the junction, and this problem has a trivial solution.

8. THE RING SOURCE AND ITS CIRCUIT REPRESENTATION

(8a) Description of the Source and Green's Theorem

A more easily realized source than the uniform discs over the cross section used in Section 3 is a thin ring around the cylinder, extending a distance ℓ along the cylinder, which produces heat uniformly over its area. Such a source is closely approximated by a thin resistive layer spread on the surface of the cylinder, through which electric current passes. The current may be steady or have any frequency. In the latter case the heat produced is the superposition of a steady source and a sinusoidally alternating one (going alternately positive and negative) as shown by (8.21) below.

The ring source will generate higher modes as well as the principal mode , but these will damp out in a short distance leaving only the principal mode. It is of interest to calculate the magnitude and phase of this principal mode part for a given strength ring source and compare it with the idealized disc type source. Also the range and strength of the next higher modes are useful to know in order to estimate the isolation distance of the source, beyond which it may be treated simply as a generator of principal mode waves.



The temperature field produced by a steady oscillating ring source $Qe^{-i\omega t}$ (so that $(Q/2\pi R\ell)e^{-i\omega t}$ is the heat produced per unit area of the ring) is easily found from the appropriate Green's function $G(\vec{r},\vec{r}')$ for a heat source in a circular cylinder. Applying Green's theorem to the functions $T(\vec{r})$ and $G(\vec{r},\vec{r}')$ over the entire volume of the cylinder, where $G(\vec{r},\vec{r}')$ will be defined explicitly below, gives

$$\iiint_V [G(\vec{r},\vec{r}')(\nabla^2+\kappa^2)T(\vec{r}) - T(\vec{r})(\nabla^2+\kappa^2)G(\vec{r},\vec{r}')]dV = \iint_S \left(G\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}\right)dS \quad . \qquad (8.1)$$

Here n is the outward normal, V and S the volume and surface area of the cylinder, ∇^2 the Laplacian in the coordinates $\vec{r}$, and the infinite extension in the z direction causes no trouble because all functions are exponentially damped away from the source.

(8b) Determination of the Green's Function.

Now choose $G(\vec{r},\vec{r}')$ so that it satisfies the wave equation in $\vec{r}$ everywhere except at $\vec{r}=\vec{r}'$, where it satisfies a source equation. (The primed variables are not to be confused with their previous use for the second rod. Only one rod is considered here.)

Since we shall be interested only in functions independent of azimuth, i.e., circularly symmetric about the z axis, we choose the source to be a ring source concentrated around a ring at particular values of r' and z', of such strength that integrating it over the entire volume gives unity. Thus, for the source at r', z' let $S(r,z,r',z')$ be the value at r,z, and

$$\int_{-\infty}^{\infty}\int_0^R S(r,z,r',z')\,2\pi r\,dr\,dz = 1 \qquad (8.2)$$

Thus $S(r,z,r',z')$ is a δ function which may be written

$$S(r,z,r',z') = \frac{\delta(r-r')}{2\pi r}\,\delta(z-z') \tag{8.3}$$

where $\delta(r-r')$ is the ordinary Dirac delta function.

Now choose $G(\vec{r},\vec{r}')$ to satisfy

$$(\nabla^2+\mathcal{H}^2)G(\vec{r},\vec{r}') = -\frac{\delta(r-r')}{2\pi r}\,\delta(z-z') \tag{8.4}$$

And the boundary condition

$$\frac{\partial G}{\partial n} = 0 \tag{8.5}$$

Then, noting that $(\nabla^2+\mathcal{H}^2)T = 0$, (8.1) gives

$$T(r',z') = \int_S G(\vec{r},\vec{r}')\frac{\partial T}{\partial n}\,dS = 2\pi R\int_{-\infty}^{\infty} G(R,z,r',z')\left(\frac{\partial T}{\partial r}\right)_{R,z} dz \;. \tag{8.6}$$

Introduce the relation between the source and $(\partial T/\partial r)$ at $r=R$ namely

$$\frac{Q}{2\pi R\ell} = K\left(\frac{\partial T}{\partial r}\right)_{r=R}\;; \quad -\frac{\ell}{2} \le z \le \frac{\ell}{2} \tag{8.7}$$

whereas

$$\left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \quad ; \quad |z| > \ell/2 \;. \tag{8.8}$$

Thus $(\partial T/\partial n)$ is known over the entire surface, which is the reason for the choice of boundary condition (8.5) on G, and (8.6) gives directly a formula for the temperature field due to the ring source,

$$T(r',z') = \frac{Q}{K\ell}\int_{-\ell/2}^{\ell/2} G(R,z,r',z')\,dz \;. \tag{8.9}$$

Thus the problem is solved when the Green's function is known.

Now $G(r,z,r',z')$ is easily found as a mode expansion, if we note that it satisfies the ordinary wave equation except at $r=r'$, $z=z'$. Hence we put

$$G(r,z,r',z') = \sum_{n=0}^{\infty} J_0(\gamma_n r)\, e^{i\kappa_n|z-z'|} f_n(r',z') \tag{8.10}$$

where the z dependence is chosen so that only outgoing waves are produced. (8.10) satisfies (8.4) term by term, except at $z=z'$. However, if (8.4) is integrated over an infinitesimal range of z about z', we obtain an expression for the discontinuity of $(\partial G/\partial z)$ at $z=z'$, on assuming that G and its derivatives with respect to r are continuous at $z=z'$. Thus

$$\left(\frac{\partial G}{\partial z}\right)_{z'+0} - \left(\frac{\partial G}{\partial z}\right)_{z'-0} = -\frac{\delta(r-r')}{2\pi r}\,. \tag{8.11}$$

Putting (8.10) in (8.11) gives

$$\sum_{n=0}^{\infty} J_0(\gamma_n r)\, 2i\kappa_n f_n(r',z') = \frac{-\delta(r-r')}{2\pi r}\,. \tag{8.12}$$

Multiplying by $rJ_0(\gamma_n r)dr$ and integrating from 0 to R gives

$$f_n(r',z') = J_0(\gamma_n r')\big/ -2\pi i\kappa_n R^2 J_0^2(\gamma_n R)\,. \tag{8.13}$$

Thus we see that $f_n(r',z')$ is really independent of z'. Putting it in (8.10) gives the desired result for G, namely

$$G(r,z,r',z') = \frac{1}{2\pi R^2}\sum_{n=0}^{\infty} \frac{J_0(\gamma_n r)J_0(\gamma_n r')}{-i\kappa_n J_0^2(\gamma_n R)}\, e^{i\kappa_n|z-z'|}\,. \tag{8.14}$$

(8c) <u>The Explicit Form of the Temperature Field and Comparison with Field of Disc Source</u>

Now that (8.14) gives the Green's function, we can find $T(r',z')$ by putting it in (8.9) to obtain

$$T(r',z') = \frac{Q}{2\pi K\ell R^2}\sum_{n=0}^{\infty}\frac{J_0(\gamma_n r')}{-i\kappa_n J_0(\gamma_n R)}\begin{cases} \dfrac{2\sin\frac{\kappa_n \ell}{2}\, e^{i\kappa_n|z'|}}{\kappa_n}\,; & |z'| > \ell/2 \quad (8.15)(a) \\[2ex] \dfrac{2(1-e^{i\kappa_n \ell/2}\cos\kappa_n z')}{-i\kappa_n}\,; & -\frac{\ell}{2} \le z' \le \frac{\ell}{2} \quad (8.15)(b) \end{cases}$$

Thus the magnitude of the outgoing lowest mode wave (and all other modes) beyond the source region is explicitly determined.

Beyond the edge of the source, i.e., for $|z'| > \ell/2$, the principal mode consists of an outward propagating wave of magnitude

$$T_p(r';z') = \frac{Q}{-2\pi R^2 K i \mathcal{H}} \cdot \frac{2\sin(\mathcal{H}\ell/2)}{\mathcal{H}\ell} e^{i\mathcal{H}z'} = Z_0\left(\frac{Q}{2}\right)\cdot\frac{2\sin(\mathcal{H}\ell/2)}{\mathcal{H}\ell} e^{i\mathcal{H}z'} = V(z') \qquad (8.16)$$

and the corresponding current of the equivalent transmission line is

$$I(z') = \frac{1}{i\mathcal{H}Z_0}\left(\frac{\partial T}{\partial z}\right)_p = \frac{Q}{2}\,\frac{\sin(\mathcal{H}\ell/2)}{(\mathcal{H}\ell/2)} e^{i\mathcal{H}z'} \qquad (8.17)$$

Thus we see that a fraction $\sin(\mathcal{H}\ell/2)/(\mathcal{H}\ell/2)$ of the heat of the source goes into the principal mode waves propagating in the two directions away from the source. This fraction, which is a typical interference factor, is very near unity when $\mathcal{H}\ell << 1$ i.e., for long wavelengths compared to the extent of the ring, as will be true for most practical cases.

For a simple disc source of the kind considered in Section 3, generating heat Q ergs per second uniformly over the cross section, only principal mode waves are set up, and the magnitude is given by

$$T(r';z') = \frac{Q}{2} Z_0 e^{i\mathcal{H}z'} \qquad (8.18)$$

This differs only by the factor $\sin(\mathcal{H}\ell/2)/(\mathcal{H}\ell/2)$ in phase and amplitude from the ring source. For all practical purposes, the ring source is thus equivalent to the disc source for production of the temperature wave. However there are also higher mode waves generated by the former, and it is desirable to get beyond the range of these. The ratio of the amplitude of the first higher mode wave in (8.15) compared to the principal mode is

$$\frac{J_0(\gamma_1 r')}{J_0(\gamma_1 R)}\,\frac{\mathcal{H}}{\mathcal{H}_1}\,\frac{\sin(\mathcal{H}_1\ell/2)/(\mathcal{H}_1\ell/2)}{\sin(\mathcal{H}\ell/2)/(\mathcal{H}\ell/2)} e^{i(\mathcal{H}_1-\mathcal{H})z} \cong \frac{2R}{\lambda}\,\frac{\sinh\frac{\pi\ell}{2R}}{\frac{\pi\ell}{2R}} e^{-\frac{\pi z}{R}} \qquad (8.19)$$

where the approximate value is obtained on putting $r'=R$, $\mathcal{H} \cong 2\pi/\lambda$, $\mathcal{H}_1 \cong i\gamma_1 \cong \frac{i\pi}{R}$ and neglecting $\mathcal{H}\ell$ and $\mathcal{H}z$. Thus if $\ell = z = R$, the ratio is $.12R/\lambda \cong .01$ to $.001$ for reasonable values of R and λ, and thus can be safely neglected at a distance of one radius from the source.

Finally we note that the ratio of the temperature (principal mode part) at position z' to the instaneous heat flux at the source is

$$\frac{T_p(z')e^{-i\omega t}}{Q\,e^{-i\omega t}} = \frac{Z_0}{2}\,\frac{\sin\mathcal{H}\ell/2}{\mathcal{H}\ell/2} e^{i\mathcal{H}z'} \cong \frac{e^{-|\mathcal{H}|z'}}{2\pi R^2 K|\mathcal{H}|} e^{i\left(|\mathcal{H}|z'+\frac{\pi}{4}-\frac{|\mathcal{H}|^2\ell^2}{24}\right)} \qquad (8.20)$$

where $|\mathcal{H}| = \sqrt{\omega/D}$, and the approximation $|\mathcal{H}|\ell << 1$ has been used. This gives the relative phase of the temperature at a point z' compared with the phase of the source (which is the phase of the second harmonic of the electric current producing the heat of the source). The principal term close to the source is the 45° lead of the temperature . At $z'=0$ it gives the thermal input impedance of the infinite rod to the ring source.

(8d) The Field of the Static Ring Source.

The above analysis using the Green's function method may also be applied solve the problem of the static ring source. This is of interest since the method of electrical heating produces such a static source superposed on the dynamic of periodic source. Thus, if the electric current is given by $J\cos\omega_o t$ feeding into a resistance R_o, the instantaneous heat produced is

$$R_o J^2\cos^2\omega_o t = \frac{R_o J^2}{2}(1+\cos 2\omega_o t) = Q_1 + Re\ Q_o e^{-i2\omega_o t} \qquad (8.21)$$

where Q_1 is a steady heat source and Q_o the amplitude of the periodic heat source with frequency double the driving electric current.

The static Green's function $G_s(\vec{r},\vec{r}')$ will be defined as a solution of the inhomogeneous Laplace equation, in place of (8.4),

$$\nabla^2 G = -\frac{\delta(r-r')}{2\pi r}\delta(z-z') \qquad (8.22)$$

with boundary condition, as in (8.5)

$$\frac{\partial G}{\partial n} = 0 \qquad (8.23)$$

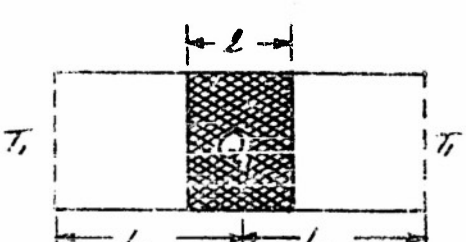
Then, just as in (8.10) - (8.14), but using static modes in the expansion

$$G_s(r,z,r',z') = C + |z-z'| f_o(r',z') + \sum_{n=1}^{\infty} J_0(\gamma_n r)e^{-\gamma_n|z-z'|} f_n(r',z')$$

$$= C + \frac{1}{2\pi R^2}\left[-|z-z'| + \sum_{n=1}^{\infty}\frac{J_0(\gamma_n r)J_0(\gamma_n r')}{\gamma_n J_0^2(\gamma_n R)}e^{-\gamma_n|z-z'|}\right] \qquad (8.24)$$

where the constant C is arbitrary. Note that the linear term in z follows from the lowest mode term of (8.14) in the limit $K \to 0$.

Now let the lowest mode part of $T(r,z)$ for $|z| > l/2$, hence outside the source, be $A z + B$, as in section 7 and consider a length 2L of rod, the ends maintained at temperature T_1 and the middle having the steady ring source Q_1. Then A and B are determined by



$$AL + B = T_1 \qquad (8.25)$$

$$\frac{Q_1}{2} = -K\pi R^2 A \qquad (8.26)$$

where (8.26) follows because only the principal mode carries heat across any cross section.

Now apply (8.1) to G_s and T within the rod of length 2L, to give

$$T(r',z') = \int_{-\ell/2}^{\ell/2} G_s(R,z,r',z')\left(\frac{\partial T}{\partial r}\right)_{R,z} 2\pi R\,dz + \int_0^R G_s(r,L,r',z')\left(\frac{\partial T}{\partial z}\right)_{r,L} 2\pi r\,dr$$

$$+\int_0^R G_s(r,-L,r',z')\left(-\frac{\partial T}{\partial z}\right)_{r,-L} 2\pi r\,dr - \int_0^R T_1\left(\frac{\partial G_s}{\partial z}\right)_{r,L} 2\pi r\,dr - \int_0^R T_1\left(-\frac{\partial G_s}{\partial z}\right)_{r,-L} 2\pi r\,dr \qquad (8.27)$$

Putting

$$\left(\frac{\partial T}{\partial r}\right)_{R,z} = \frac{Q_1}{2\pi R\ell K} \text{ for } -\frac{\ell}{2} \le z \le \frac{\ell}{2} \;;\; \left(\frac{\partial T}{\partial z}\right)_{r,L} = A = \frac{-Q_1}{2K\pi R^2} = -\left(\frac{\partial T}{\partial z}\right)_{r,-L}$$

and using only the lowest mode parts of G and $\left(\frac{\partial G_s}{\partial z}\right)$ at $z = \pm L$ gives

$$T(r',z') = \int_{-\ell/2}^{\ell/2}\left(C - \frac{(z'-z)}{2\pi R^2} + \frac{1}{2\pi R^2}\sum_{n=1}^{\infty}\frac{J_0(\gamma_n r')}{\gamma_n J_0(\gamma_n R)}e^{-\gamma_n(z'-z)}\right)\frac{Q_1}{2\pi R\ell K}\cdot 2\pi R\,dz$$

$$+\int_0^R\left(C - \frac{(L-z')}{2\pi R^2}\right)\left(\frac{-Q_1}{2\pi K R^2}\right)2\pi r\,dr + \int_0^R\left(C - \frac{(z'+L)}{2\pi R^2}\right)\left(-\frac{Q_1}{2\pi K R^2}\right)2\pi r\,dr \qquad (8.28)$$

$$-\int_0^R T_1\left(-\frac{1}{2\pi R^2}\right)2\pi r\,dr - \int_0^R T_1\left(-\frac{1}{2\pi R^2}\right)2\pi r\,dr \;;\; z' > \ell/2 .$$

Hence the solution to the static problem is

$$T(r',z') = T_1 - \frac{Q_1}{2\pi R^2 K}\left(z' - L - \sum_{n=1}^{\infty}\frac{J_0(\gamma_n r')}{\gamma_n J_0(\gamma_n R)}e^{-\gamma_n z'}\frac{\sinh(\gamma_n \ell/2)}{(\gamma_n \ell/2)}\right);\; z' > \frac{\ell}{2} . \qquad (8.29)$$

Evidently (8.29) has the correct linear behavior in the lowest mode, and agrees with (8.15 a) when κ_n there is replaced by $i\gamma_n$ (Note that the arbitrary constant C has cancelled out of the result.)

Appendix I — LIST AND DEFINITIONS OF SYMBOLS IN ORDER OF FIRST APPEARANCE

(Cross references are given for cases of repeated or nearly identical symbols)

1) $\vec{Q}$ vector heat flux ($\text{joules}/\text{cm}^2\,\text{sec}$) (cf. 17, 20, 47, 70)

2) $T(\vec{r}, t)$ temperature field as a function of vector position and time (degrees centigrade) (cf. 3, 16, 21, 46)

3) K thermal conductivity ($\text{joules}/\text{cm sec deg}$)

4) ρ density (gm/cm^3)

5) C_g specific heat ($\text{joules}/\text{gm deg}$) (cf. 36, 59, 55)

6) D diffusivity $= (K/\rho C_g)(\text{cm}^2/\text{sec})$ (cf. 49)

7) ω angular frequency $= 2\pi\nu$

8) r, z cylindrical radial and axial coordinates (cf. 60)

9) $T^{(0)}(\vec{r})$ space part of harmonically varying complex temperature $T^{(0)}(\vec{r})e^{-i\omega t}$ (the superscript is dropped in later sections) (cf. 2)

10) $\varkappa, k$ principal mode propagation constant $= (1+i)(\omega/2D)^{1/2} = (1+i)k$ where k is the real wave number (cf. 11)

11) $\varkappa_n$ nth mode (complex) propagation constant in factor $e^{\pm i\varkappa_n z}$, $\varkappa_n^2 = \varkappa^2 - \gamma_n^2$ (cf. 10)

12) γ_n a (real) radial "wave number" determined by the condition $J_1(\gamma_n R) = 0$ (cf. 51, 81)

13) $J_0(\gamma_n r), J_1(\gamma_n r)$ Bessel functions of order 0 and 1 of radial coordinate

14) β_n nth root of $J_1(x) = 0$, $n = 0, 1, 2, \cdots$; $\gamma_n R = \beta_n$ (cf. 32, 81)

15) R radius of circular cylindrical rod (cf. 37, 75)

16) T_1, T_2 steady temperatures used as boundary conditions (for static heat flux problems) (cf. 2)

17) Q_1 magnitude of steady heat flux due to a static source (cf. 1, 20)

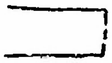
18) A cross section area of general cylindrical rod (cf. 31, 44)

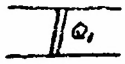
19) l, l_1, l_2 lengths of rod in heat flow problems (cf. 69)

20) Q_0 amplitude of harmonically varying source $Q_0 e^{-i\omega t}$ (cf. 1, 17)

21) $T^{(1)}, T^{(0)}$ static and dynamic temperatures, solutions of static and dynamic heat flow problems respectively; total $T = T^{(1)} + T^{(0)}e^{-i\omega t}$ (cf. 2, 9)

22)  representation of rod terminating in free end; the free surfaces are all solid lines

23)  representation of rod with constant temperature end at static value T_2

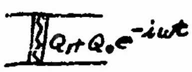
24) 

representation of steady heat source Q_1 , disc type

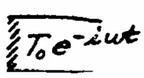
25)  representation of dynamic boundary temperature zero on end of rod

26) 

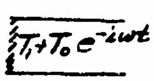
representation of dynamic disc type source $Q_0 e^{-iwt}$

27) 

representation of combined static and dynamic source

28) 

representation of oscillating temperature boundary condition

29) 

representation of combined steady plus oscillating temperature boundary condition

30)  representation of semi-infinite extension of rod to the right

31) A, B amplitudes of travelling principal mode waves in $+z$ and $-z$ directions respectively (cf. 18, 44, 56)

32) α, β amplitudes of cosine and sine standing principal mode temperature waves (cf. 14, 81)

33) $V(z), I(z)$ (complex) voltage and current on equivalent transmission line (cf. 34, 38, 40)

34) V_0, I_0 values of $V(0)$ and $I(0)$, hence usually refer to reference plane (cf. 33, 40)

35) Z_0, Y_0 characteristic impedance and admittance of equivalent transmission line (cf. 39, 41, 42, 43, 48)

36) C constant factor introduced in defining $V(z) = CT(z)$, later taken as unity (cf. 5, 50, 55)

37) $\mathcal{R}$ complex voltage (or temperature) reflection coefficient at a termination (cf. 15, 75)

38) $A', B', \alpha', \beta', V_0', I_0'$, etc. quantities referring to rod II, as opposed to rod I, in general the prime will be used for this purpose (cf. 31, 32, 33, 34, 35, 36, etc.)

39) $Z_{11}, Z_{12}, Z_{21}, Z_{22}$ elements of impedance matrix relating V_0, I_0, V_0', I_0' (cf. 35, 41, 42, 43)

40) V_0'', I_0'' voltage and current defined in rod II with opposite sign convention for current (i.e. $V_0'' = V_0'$, $I_0'' = -I_0'$) (cf. 33, 34)

41) $Z_{11}'', Z_{12}'', Z_{21}'', Z_{22}''$ impedance matrix elements relating V_o', I_o', V_o'', I_o'' (cf. 39)

42) Z_t' arbitrary impedance terminating rod II at distance l from reference plane (cf. 39, 43, 48)

43) Z' impedance of rod II seen at reference plane $\equiv (V_o'/I_o')$ (cf. 39, 42, 48)

44) A_n, A_n' amplitudes of higher modes in rods I and II, $n = 1, 2, 3 \cdots$ (cf. 18, 31)

45) $E(r)$ the function $K(\partial T/\partial z)$ in the junction plane ($z = 0$) (cf. 57)

46) $T_p(z), (\partial T/\partial z)_p$ principal mode parts of T and $(\partial T/\partial z)$ (cf. 2)

47) $Q(z)$ total (complex) heat flux through the cross section of the rod at position z (cf. 1)

48) Z series impedance representing the effect of the change of cross section on transmission lines I and II (cf. 35, 42, 43)

49) D_{lm} a numerical constant depending on (K/K'), R, R', γ_l, γ_n' but independent of frequency (cf. 6)

50) C_{lm} a numerical constant depending on sum of D_{ln}, D_{mn} for all n and on $\mathcal{K}_n'$ (hence on frequency) (cf. 5, 36, 55)

51) r the ratio R/R', always >1 by convention. (cf. 12, 81)

52) $=$ is equal to ; $\simeq$ is asymptotically equal to

$\equiv$ is identically equal to ; $\geq(\leq)$ is greater (less) than or equal to

$\cong$ is approximately equal to ; $(=)$ is formally equal to

53) S, S_n $S = \sum_{n=1}^{\infty} S_n$, $S_n = J_1^2(\beta_n/r)/\beta_n^3 J_0^2(\beta_n)$ (cf. 58, 59, 61, 65, 68, 72, 81)

54) $\Psi(z)$, $\Psi'(z), \Psi''(z)$ the psi function or the logaritmic derivative of the factorial function; derivatives of $\Psi(z)$

55) C_E Euler's constant $= .577216$ (cf. 5, 36)

56) B_n the Bernoulli numbers (cf. 31)

57) E_n the Euler numbers (cf. 45, 81)

58) $\mathcal{S}, \mathcal{S}_n$ $\mathcal{S} = \sum_{n=1}^{\infty} \mathcal{S}_n$, $\mathcal{S}_n = S_n\left[1 - \left(\frac{\mathcal{K}'R'}{\beta_n}\right)^2\right]^{-\frac{1}{2}}$ (cf. 53, 61)

59) $\tilde{S}, \tilde{S}_n$ $\tilde{S} = \sum_{n=1}^{\infty} \tilde{S}_n$, $\tilde{S}_n \simeq S_n$ thus $\tilde{S}_n$ is asymptotic to S_n for large n (cf. 53, 65)

60) z, x, y $z = 2\pi/r$, $x = z/4 = \pi/2r$, $y = \pi/2 - x = (\pi/2)(1 - r)$ notation used for convenience in the sum formulas of section 6 (cf. 9)

61) $\mathcal{J}^{(l,m)}, \mathcal{J}_n^{(l,m)}$ $\mathcal{J}^{(l,m)} = \sum_{n=1}^{\infty} \mathcal{J}_n^{(l,m)}$; $\mathcal{J}_n^{(l,m)} = \mathcal{J}_n / (1 - \frac{r^2 B_l}{B_n^2})(1 - \frac{r^2 B_m^2}{B_n^2})$
thus $\mathcal{J} = \mathcal{J}^{(0,0)}$, $\mathcal{J}_n = \mathcal{J}_n^{(0,0)}$ (cf. 53,58)

62) μ, a, b numerical constants $\mu = \sqrt{B_l}/\pi$, $a = \mu + 1/4$, $b = \mu - 1/4$ (cf. 66)

63) $G(a,z), F(a,z)$ functions of two variables defined by sums in (6.65), (6.66), (6.69) and required in asymptotic forms for C_{ol} (cf. 67)

64) $Ci(z), Si(z)$ cosine and sine integral functions $= -\int_z^{\infty}\left(\frac{\cos z}{z}\right)dz$, $\int_0^z\left(\frac{\sin z}{z}\right)dz$

65) $\tilde{S}^{(l,m)}, \tilde{S}_n^{(l,m)}$ $\tilde{S}^{(l,m)} = \sum_{n=1}^{\infty} \tilde{S}_n^{(l,m)}$, $\tilde{S}_n^{(l,m)} = \tilde{S}_n / (1 - \frac{r^2 B_l^2}{B_n^2})(1 - \frac{r^2 B_m^2}{B_n^2}) \simeq S_n^{(l,m)}$ (cf. 53,59)

66) $\tilde{\mu}, \tilde{a}, \tilde{b}$ asymptotic values of μ, a, b obtained by putting $B_l = 5\pi/4$, $\mu = 5\pi/4$ (cf. 62)

67) $G_2(a,z), F_2(a,z)$ functions of two variables defined by sums in 6.89 and required in asymptotic forms for C_{ll} (cf. 63)

68) $S_p(a), U_p(a)$ the functions defined by W. Hahn as Fourier sums, given in (6.99) (cf. 53)

69) l length of ring source (cf. 19)

70) Q strength of ring source, producing heat $Qe^{-i\omega t}$ (cf. 1,17,20)

71) $G(\bar{r},\bar{r}')$ Green's function in infinite circular cylinder with free wall boundary conditions $(\partial G/\partial n) = 0)$ (cf. 74)

72) $S(r,z,r',z')$ a ring source function giving strength at r,z, of a delta function ring at r,z. (cf. 53 etc.)

73) $\delta(r-r')$ the *Dirac* delta function (cf. 81)

74) $G_s(\bar{r},\bar{r}')$ the static Green's function corresponding to $G(\bar{r},\bar{r}')$ (cf. 71)

75) Re, Im real, imaginary parts of, $z = Re\,z + i\,Im\,z$

76) $L_n(t)$ the integral $(1/n!)\int_0^t e^{it} t^n dt$

77) $Exp_n(x)$ the incomplete exponential of x, consisting of the first n terms of the power series $\sum_{l=0}^{n-1} x^l/l!$

78) $\overline{Exp_n}(x)$ the complement of the incomplete exponential, consisting of the tail of the power series from $x^n/n!$ on

79) $[\frac{n}{2}]$ the largest integer in $n/2$

80) $Cos_n(z)$, $Sin_n(z)$, $\overline{Cos_n}(z)$, $\overline{Sin_n}(z)$ incomplete cosine and sine functions and their complements, defined in (II .8).

81) $f(z)$, a, I, I_s, h, ς, d, ρ, E_0, E_1, E_2, θ, F_s, α, β, γ, C_s, S_s, δ_{nt} are all symbols occurring with special meanings in Appendix IIb, different from the uses in the main text and defined above. For these special definitions refer to IIb.

Appendix II - EVALUATION OF INTEGRALS IN SUM FORMULAS FOR THE $C_{\ell m}$

(IIa) Power Series Expansion Method

The integrals in (6.69) may be evaluated by expanding cot $z/2$ in powers of z and integrating term by term. The resulting expansion is convergent out to $z = 2\pi$ which is, in fact, the largest value of $z = 2\pi/\gamma$ that occurs in the change of cross section problem. However, the expansion need never be used for values of z greater than π, where it is strongly convergent, since we have the identity

$$\int_{\pi}^{z} e^{iaz} \cot\frac{z}{2}\, dz = e^{2\pi a i} \int_{\pi}^{2\pi - z} e^{-iaz} \cot\frac{z}{2}\, dz \qquad (II.1)$$

For values of z greater than π which may occur in other applications (see the section on the Hahn functions) the periodicity of $\cot z/2$ (period 2π) permits reduction of

$$\int_{2m\pi}^{(2m+1)\pi} \text{ to } \int_{0}^{\pi} \quad \text{and of} \int_{(2m+1)\pi}^{(2m+2)\pi} \text{ to } \int_{\pi}^{2\pi}$$

which by (II.1) may also be reduced to $\int_0^{\pi}$.

In fact z may be reduced in the original arguments of $F(a,z)$, $G(a,z)$ by the easily derived formulas, where p is an integer,

$$F(a,z) = \cos 2\pi a p\, F(a, z - 2\pi p) - \sin 2\pi a p\, G(a, z - 2\pi p)$$

$$G(a,z) = \cos 2\pi a p\, G(a, z - 2\pi p) + \sin 2\pi a p\, F(a, z - 2\pi p) \qquad (II.1b)$$

It is also noteworthy that values of a greater than 1 need not be considered, since the original arguments in $F(a,z)$, $G(a,z)$ may be reduced by means of

$$F(a,z) = F(a-p, z) - \sum_{q=0}^{p-1} \frac{\cos(a-q)z}{a-q}$$

$$G(a,z) = G(a-p, z) - \sum_{q=0}^{p-1} \frac{\sin(a-q)z}{a-q} \qquad (II.1c)$$

It is useful to reduce z and a in this way since the convergence of the series (II.10) and (II.12) is then improved by making the quantities $\overline{Sin}_n(az)$ and $\overline{Cos}_n(az)$ decrease more rapidly with n. Also the integration range in the numerical integration method of Appendix IIb is reduced.

Thus we have

$$\frac{1}{2}\int_{\pi}^{z} e^{iaz} \cot z/2\, dz = \frac{1}{2}\int_{\pi}^{z} e^{iaz} \left(\frac{2}{z} - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} \left(\frac{z}{2}\right)^{2n-1} \right) dz \qquad (II.2)$$

$$= \left(Ci(az) - Ci(a\pi)\right) + i\left(Si(az) - Si(a\pi)\right) - \sum_{n=1}^{\infty} \frac{B_n}{(2n)!} \int_{\pi}^{z} e^{iaz} z^{2n-1}\, dz$$

The integrals in (II.2) may be explicitly evaluated in simple series form as follows

$$\frac{1}{(2n-1)!}\int_{\pi}^{z} e^{iaz} z^{2n-1} dz = \frac{1}{a^{2n}}\left[L_{2n-1}(az) - L_{2n-1}(a\pi)\right] \qquad (II.3)$$

where

$$L_n(t) \equiv \frac{1}{n!}\int_0^t e^{it} t^n dt = -ie^{it}\frac{t^n}{n!} + iL_{n-1}(t)$$
$$L_0(t) = -i(e^{it}-1) \qquad (II.4)$$

Repeated application of (II.4) yields

$$L_n(t) = i^{n+1}\left[1 - e^{it}\sum_{\ell=0}^{n} i^{-\ell}\frac{t^\ell}{\ell!}\right] = i^{n+1}\left[1 - e^{it}\,Exp_{n+1}(-it)\right] \qquad (II.5)$$

where we define the incomplete exponent $Exp_{n+1}(x)$ and its complement $\overline{Exp}_{n+1}(x)$ by

$$Exp_{n+1}(x) \equiv \sum_{\ell=0}^{n}\frac{x^\ell}{\ell!} \equiv e^x - \overline{Exp}_{n+1}(x)$$
$$\overline{Exp}_{n+1}(x) = \sum_{\ell=n+1}^{\infty}\frac{x^\ell}{\ell!} \qquad (II.6)$$

For the imaginary argument in (II.5) it is convenient to separate Exp_{n+1} into the incomplete cosine and sine functions, defined analogously by

$$Exp_{n+1}(-it) = \sum_{p=0}^{[\frac{n}{2}]}(-)^p\frac{t^{2p}}{(2p)!} - i\sum_{p=0}^{[\frac{n-1}{2}]}(-)^p\frac{t^{2p+1}}{(2p+1)!} \equiv Cos_{[\frac{n}{2}]+1}(t) - i\,Sin_{[\frac{n-1}{2}]}(t) \qquad (II.7)$$

where the notation $[\frac{n}{2}]$ means the largest integer in $n/2$.
Thus we have

$$Cos_{n+1}(x) \equiv \sum_{p=0}^{n}(-)^p\frac{x^{2p}}{(2p)!} = \cos x - \sum_{p=n+1}^{\infty}(-)^p\frac{x^{2p}}{(2p)!} \equiv \cos x - \overline{Cos}_{n+1}(x)$$

$$(II.8)$$

$$Sin_{n+1}(x) \equiv \sum_{p=0}^{n}(-)^p\frac{x^{2p+1}}{(2p+1)!} = \sin x - \sum_{p=n+1}^{\infty}(-)^p\frac{x^{2p+1}}{(2p+1)!} \equiv \sin x - \overline{Sin}_{n+1}(x)$$

Putting (II.5) into (II.3) gives for the integrals in (II.2)

$$\frac{1}{(2n-1)!}\int_{\pi}^{z} e^{iaz} z^{2n-1} dz = \frac{(-)^{n+1}}{a^{2n}}\left[e^{iaz}\,Exp_{2n}(-iaz) - e^{ia\pi}\,Exp_{2n}(-ia\pi)\right].$$

Finally we can write the real and imaginary parts of (II.2) which occur respectively in $F(b,z)$ and $G(b,z)$, in the forms (II.9) and (II.11),

$$\frac{1}{2}\int_{\pi}^{z}\cos az\cot\frac{z}{2}\,dz = Ci(az) - Ci(a\pi) \tag{II.9}$$

$$+\sum_{n=1}^{\infty}(-)^n\frac{B_n}{2na^{2n}}\Big[\cos az\, Cos_n(az) + \sin az\, Sin_n(az) - \cos a\pi\, Cos_n(a\pi) - \sin a\pi\, Sin_n(a\pi)\Big]$$

Since the functions Cos_n and Sin_n do not become small as $n \to o$ but approach cos and sin, respectively, it is best to rewrite (II.9) in terms of the complementary functions, which go rapidly to zero.

$$\frac{1}{2}\int_{\pi}^{z}\cos az\cot\frac{z}{2}\,dz = Ci(az) - Ci(a\pi) \tag{II.10}$$

$$+\sum_{n=1}^{\infty}(-)^n\frac{B_n}{2na^{2n}}\Big[-\cos az\,\overline{Cos}_n(az) - \sin az\,\overline{Sin}_n(az) + \cos a\pi\,\overline{Cos}_n(a\pi) + \sin a\pi\,\overline{Sin}_n(a\pi)\Big]$$

Similarly

$$\frac{1}{2}\int_{\pi}^{z}\sin az\cot\frac{z}{2}\,dz = Si(az) - Si(a\pi) - \sum_{n=1}^{\infty}\frac{B_n(-)^{n+1}}{2na^{2n}}\Big[\sin az\, Cos_n(az) - \cos az\, Sin_n(az) \tag{II.11}$$
$$-\sin a\pi\, Cos_n(a\pi) + \cos a\pi\, Sin_n(a\pi)\Big]$$

$$= Si(az) - Si(a\pi) + \sum_{n=1}^{\infty}\frac{(-)^n B_n}{2na^{2n}}\Big[-\sin az\,\overline{Cos}_n(az) + \cos az\,\overline{Sin}_n(az) + \sin a\pi\,\overline{Cos}_n(a\pi) - \cos a\pi\,\overline{Sin}_n(a\pi)\Big] \tag{II.12}$$

(IIb) **Numerical Integration Method and Filon's Formula**

The evaluation of the integrals in (6.69) or, better, of the integral in (6.71)

$$\int_{z}^{\pi} e^{iaz}\left(\frac{1}{z} - \frac{1}{2}\cot\frac{z}{2}\right)dz \equiv \int_{z}^{\pi} e^{iaz} f(z)\,dz \tag{II.13}$$

may be simply and accurately done by direct numerical integration, even for large values of "a", provided we make use of Filon's method[12] which treats the trigonometric factors analytically. The method is based on Simpson's rule, which requires a tabulation of the integrand fine enough so that the integrand may be represented to the desired accuracy by a parabola through each consecutive triplet of points.

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[12] L. N. G. Filon, Proc. Roy. Soc. Edin. 49, 38-46 (1928-9)
See also C. J. Tranter "Integral Transforms in Mathematical Physics" Methuen 1951, p. 67.

Accordingly, if this tabulation is done for $f(z)$, the integral (II.13) may be evaluated as accurately as Simpson's rule applied to $f(z)$ alone, using the same set of points, by doing analytically the integral of the product of e^{iaz} and the approximating parabola. The analysis is repeated here briefly for completeness and ease of reference, and slightly extended.

Consider

$$I \equiv \int_c^d e^{iaz} f(z)\,dz \equiv \sum_{t=0}^{n-1} I_{2t+1} \equiv \sum_{t=0}^{n-1} \int_{c+2th}^{c+(2t+2)h} e^{iaz} f(z)\,dz \equiv \sum_{t=0}^{n-1} \int_{z_{2t}}^{z_{2t+2}} e^{iaz} f(z)\,dz \qquad (II.14)$$

where the integration range c to d has been divided into $2n$ parts each of width h and the notation

$$z_s = c+sh \quad , \quad s=0,\cdots 2n \qquad (II.15)$$

has been introduced for the s^{th} abscissa. The integral is split into n sub-integrals, and the subintegral from z_{2t} to z_{2t+2} is called I_{2t+1}.

Let the subdivision of $(d-c)$ be fine enough so that in each of the subintegrals $f(z)$ is represented to the desired accuracy by a parabola, hence in I_{2t+1} we may use the three point Lagrangian formula for $f(z)$,

$$f(c+(s+p)h) \equiv f_{s+p} = \frac{p(p-1)}{2} f_{s-1} - (p^2-1) f_s + \frac{p(p+1)}{2} f_{s+1} \qquad (II.16)$$

$$s = 2t+1 \, , \quad -1 \leqslant p \leqslant 1$$

In (II.16) the notation

$$f_s \equiv f(z_s) \equiv f(c+sh) \qquad (II.17)$$

has been used, and p is the usual Lagrangian coordinate denoting a running fraction of the basic tabular interval h.

Then

$$I_s = \int_{-1}^{1} e^{ia[c+(s+p)h]} \left\{ \frac{(p^2-p)}{2} f_{s-1} - (p^2-1) f_s + \frac{(p^2+p)}{2} f_{s+1} \right\} h\,dp \qquad (II.18)$$

$$= e^{ia[c+sh]} h \left\{ \frac{(E_2-E_1)}{2} f_{s-1} - (E_2-E_0) f_s + \frac{(E_2+E_1)}{2} f_{s+1} \right\} \qquad (II.19)$$

where

$$\left.\begin{aligned} E_0 &= \int_{-1}^{1} e^{i\theta p}\,dp = \frac{2\sin\theta}{\theta}\ ; \quad \theta = ah \\ E_1 &= \int_{-1}^{1} e^{i\theta p}\,p\,dp = \frac{-2i\cos\theta}{\theta} + \frac{2i\sin\theta}{\theta^2} \\ E_2 &= \int_{-1}^{1} e^{i\theta p}\,p^2\,dp = \frac{2\sin\theta}{\theta} + \frac{4\cos\theta}{\theta^2} - \frac{4\sin\theta}{\theta^3} \end{aligned}\right\} \qquad (II.20)$$

Putting (II.20) into (II.19) and using the notation

$$F_s \equiv F(z_s) \equiv e^{iaz_s} f(z_s) \equiv e^{ia[c+sh]} f(c+sh) \qquad (II.21)$$

for the complete integrand in (II.13), we have

$$I_s = h\left[\left(\frac{\beta}{2}+i\alpha\right)F_{s-1} + \gamma F_s + \left(\frac{\beta}{2}-i\alpha\right)F_{s+1}\right] \qquad (II.22)$$

where

$$\left.\begin{aligned} \alpha &\equiv \frac{\sin\theta E_2}{2} + i\,\frac{\cos\theta E_1}{2} = \frac{1}{\theta} + \frac{\sin\theta\cos\theta}{\theta^2} - \frac{2\sin^2\theta}{\theta^3} \\ \beta &\equiv \cos\theta E_2 - i\sin\theta E_1 = \frac{2}{\theta^2}(1+\cos^2\theta) - \frac{4\sin\theta\cos\theta}{\theta^3} \\ \gamma &\equiv E_0 - E_2 = \frac{4\sin\theta}{\theta^3} - \frac{4\cos\theta}{\theta^2} \end{aligned}\right\} \qquad (II.23)$$

and α, β, γ are the same quantities used by Filon or Tranter.

Taking the sum of $I_s = I_{2t+1}$ over all the subintegrals, we have for the entire integral

$$I = \sum_{t=0}^{n-1} I_{2t+1} = h\left[\beta \sum_{t=0}^{n}\left(F_{2t} - \frac{(\delta_{0t}+\delta_{nt})}{2}\right) + \gamma \sum_{t=0}^{n-1} F_{2t+1} + i\alpha(F_0 - F_{2n})\right]. \qquad (II.24)$$

(II.24) is the desired integration formula, where the first sum is over the integrands at all even abscissas minus one half the first and last terms ($\delta_{0t} = 0$ for $t \neq 0$, $= 1$ for $t = 0$, is the Kronecker delta symbol), and the second sum is over the integrands at all odd abscissas.

In taking the real and imaginary parts of (II.24) to evaluate the integral with cosine and sine factors, the last term with the coefficient $i\alpha$ mixes the two types of integrands. Thus putting

$$F_s \equiv e^{iaz_s} f(z_s) \equiv \cos az_s\, f(z_s) + i\sin az_s \equiv C_s + i\,S_s \qquad (II.25)$$

we have

$$\int_c^d \cos az\, f(z)\,dz = h\Big[\beta\sum_{t=0}^{n}\Big(C_{2t} - \frac{\delta_{0t}+\delta_{nt}}{2}\Big) + \gamma\sum_{t=0}^{n-1} C_{2t+1} - \alpha(S_0 - S_{2n})\Big] \qquad (II.26)$$

$$\int_c^d \sin az\, f(z)\,dz = h\Big[\beta\sum_{t=0}^{n}\Big(S_{2t} - \frac{\delta_{0t}+\delta_{nt}}{2}\Big) + \gamma\sum_{t=0}^{n-1} S_{2t+1} + \alpha\,(C_0 - C_{2n})\Big] \qquad (II.27)$$

Finally, we note that if θ is small, (II.23) is inconvenient for calculation of α, β, γ because the difference of large nearly equal terms occurs. As Tranter indicates, the power series expansions are convenient in this case, and we may write these in the general forms

$$\alpha = \sum_{n=0}^{\infty} (-)^n \frac{(2n+2)}{(2n+6)!} 2^{2n+4} \theta^{2n+3}$$

$$\beta = \sum_{n=0}^{\infty} (-)^{n+1} \frac{(2n-1)}{(2n+3)!} 2^{2n+2} \theta^{2n} \qquad (II.28)$$

$$\gamma = \sum_{n=0}^{\infty} (-)^n \frac{4(2n+2)}{(2n+3)!} \theta^{2n}$$

The numerical values of the coefficients are tabulated below and suffice in, at most, ten terms to calculate α, β, γ to 6 decimals for $0 \le \theta \le 2$. At most six terms are needed for $\theta \le 1$. For $\theta > 1$ no numerical difficulties arise in using (II.23).

In applying (II.24) or (II.26) and (II.27) to the integral (II.13), we have merely to tabulate $f(z) = 1/z - 1/2 \cot z/2$ at small enough intervals over the range of interest $(0 \text{ to } \pi)$, that $f(z)$ can be well represented by a parabola over these intervals. Otherwise stated, only the second differences of the tabulated values of $f(z)$ are then needed in interpolating $f(z)$ within the range. As shown by the values below, if we choose the points $z = n\pi/20$ so that $h = \pi/20 = .157080$, then the third and fourth differences contribute at most 2 in the 6th decimal, and that only near $z = \pi$.

Thus the procedure is this. Put $\int_z^{\pi} = \int_{z_0}^{\pi} + \int_{z}^{z_0}$ where z_0 is the nearest even multiple of $\pi/20$ to z. Then if, for example, the integral contains the cosine, as in (II.26), tabulate $\cos az_s\, f(z_s)$ between z_0 and π, and also $S_0 = \sin az_0\, f(z_0)$ and $S_{2n} = \sin a\pi\, f(\pi)$. The two sums in (II.26) containing, at most, some ten terms each, can now be added up. Finally find $\theta = (a\pi/20)$ and α, β, γ and substitute in (II.26). This gives

$$\int_{z_0}^{\pi} \cos az\, f(z)\, dz .$$

The end correction $\int_{z_0}^{z} \cos az\, f(z)\,dz$ requires a second application of (II.26), or better of (II.22) since only one parabolic section is needed ($z - z_0 < \pi/20$ so that $f(z)$ is parabolic over the whole range z_0 to z.) For this case we have $h = (z-z_0)/2$, $\theta = a(z-z_0)/2$ both different from the main integral, and $F_s = F\left(\frac{z_0+z}{2}\right)$; $F_{s-1} = F(z_0)$; $F_{s+1} = F(z)$. $f(z)$ and $f\left(\frac{z+z_0}{2}\right)$ are found by interpolation from the table of $f(z)$, using second differences.

The work involved in tabulating (II.13) by direct integration is less, in general, than in using formulas (II.10) and (II.12) because of the labor in evaluating the sequences of the four coefficients $\overline{Cos}_n(az)$, $\overline{Sin}_n(az)$, $\overline{Cos}_n(a\pi)$, $\overline{Sin}_n(a\pi)$. Each of these must be found from a separate series, whereas the terms in (II.26) require only tabulation of $\cos az$ (or $\sin az$) and multiplication by the known values of $f(z)$. (Of course α, β, γ must also be evaluated, and the end correction found). However, if more decimal places were required in the result, direct integration would require more tabulation of $f(z)$ and more terms in the sums, so that eventually the series (II.10) and (II.12) would be preferable. Six decimal places should be ample for any practical needs, however.

TABLE OF COEFFICIENTS IN EXPANSION OF α, β, γ

n	(α) $(-)^n \frac{(2n+2)\,2^{2n+4}}{(2n+6)!}$		(β) $(-)^{n+1} \frac{(2n-1)\,2^{2n+2}}{(2n+3)!}$		(γ) $(-)^n \frac{4(2n+2)}{(2n+3)!}$
0	.044 444	4	.666 666	7	1.333 3333
1	−.006 349	21	.133 333	3	−.133 3333
2	.000 423	280	−.038 095	24	.004 76190
3	−.000 017	102	.003 527	337	−.000 088183
4	.000 000	4698	−.000 279	573	.000 001002
5	−.000 000	00940	.000 005	9200	−.000 0000077
6	.000 000	00014	−.000 000	1378	.000 00000004
7	−.000 000	000 002	.000 000	002 40	
8			−.000 000	000 0323	
9			.000 000	000 000 35	

TABLE OF VALUES OF $f(z) = \frac{1}{z} - \frac{1}{2}\cot\frac{z}{2}$ FOR NUMERICAL INTEGRATION OF $\int_z^{\pi} e^{iaz} f(z)\,dz$, $0 \le z \le \pi$

n	$z = \frac{n\pi}{20}$	$f(z) = \frac{1}{z} - \frac{1}{2}\cot\frac{z}{2}$	$M'' = \Delta'' - .184\Delta^{iv}$
0	0	0	0
1	.157 080	.013 095	33
2	.314 159	.026 223	65
3	.471 239	.039 416	97
4	.628 319	.052 707	135
5	.785 398	.066 133	168
6	.942 478	.079 728	208
7	1 .099 558	.093 532	247
8	1 .256 637	.107 584	293
9	1 .413 717	.121 930	343
10	1 .570 796	.136 620	394
11	1 .727 876	.151 705	454
12	1 .884 956	.167 245	521
13	2 .042 035	.183 308	592
14	2 .199 115	.199 965	681
15	2 .356 195	.217 306	778
16	2 .513 274	.235 428	889
17	2 .670 354	.254 442	1026
18	2 .827 434	.274 486	1176
19	2 .984 513	.295 712	1365
20	3 .141 593	.318 310	1586

Interpolation formula (Everett)

$$g(z_0+ph) = q\,g_0 + p\,g_1 + E_0'' M_0'' + E_1'' M_1''$$

$$E_0'' = -\tfrac{1}{6}\,q(q^2-1) \quad , \quad g_0 = g(z_0)$$

$$E_1'' = -\tfrac{1}{6}\,p(p^2-1) \quad , \quad g_1 = g(z_0+h)$$

Use of this formula with the tabulated modified second differences M'' gives six decimals in $f(z)$ over entire range of $0 \le z \le \pi$.

Neglect of third differences makes a maximum error of 2 in the sixth place; neglect of fourth differences a maximum error of 1 in the sixth place.

Appendix III.

Numerical example of summation of asymptotic form of the higher order term $\tilde{\zeta}^{(0,1)}$.

From (6.60)

$$\tilde{\zeta}^{(0,1)} = \frac{\gamma}{2\pi^3}\sum_{n=1}^{\infty}\frac{(1-\sin(n+1/4)z)}{(n+1/4)^3\left(1-\frac{\tilde{\rho}^2}{(n+1/4)^2}\right)} \qquad \text{where } \tilde{\rho}=\frac{5\gamma}{4},\ z=\frac{2\pi}{\gamma}$$

Hence by (6.83)

$$\tilde{\zeta}^{(0,1)} = \frac{\gamma}{4\pi^3\tilde{\rho}^2}\left[-\Psi(\tilde{a})-\Psi(\tilde{b})+2\Psi(\tfrac{1}{4})+\frac{1}{\tilde{b}}+F(\tilde{a},z)-F(\tilde{b},z)-\frac{\cos\tilde{b}z}{\tilde{b}}+2G(\tfrac{1}{4},z)\right]$$

where by (6.65)

$$G(\tfrac{1}{4},z)=\frac{\pi}{2}-4\sin(\tfrac{z}{4})+\ln 2-\ln(\tfrac{\pi}{2}-\tfrac{z}{4})-\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n)!}B_n\left(\frac{\pi}{2}-\frac{z}{4}\right)^{2n},\quad \tilde{a}=\tilde{\rho}+\tfrac{1}{4},\ \tilde{b}=\tilde{\rho}-\tfrac{1}{4}$$

$$F(\tilde{b},z)=\cos\tilde{b}\pi\left[-\tfrac{1}{2}\Psi\left(\frac{\tilde{b}-2}{2}\right)+\tfrac{1}{2}\Psi\left(\frac{\tilde{b}-1}{2}\right)\right]-\frac{1}{2\tilde{b}}\left[\cos\tilde{b}z+\cos\tilde{b}\pi\right]-\frac{1}{2}\int_{\pi}^{z}\cos\tilde{b}z\cot\frac{z}{2}\,dz$$

and by (II.10)

$$\frac{1}{2}\int_{\pi}^{z}\cos\tilde{b}z\cot\frac{z}{2}\,dz = Ci(\tilde{b}z)-Ci(\tilde{b}\pi)+\sum_{n=1}^{\infty}\frac{(-)^nB_n}{2n\tilde{b}^{2n}}\Big[-\cos\tilde{b}z\,\overline{Cos}_n(\tilde{b}z)-\sin\tilde{b}z\,\overline{Sin}_n(\tilde{b}z)$$
$$+\cos\tilde{b}\pi\,\overline{Cos}_n(\tilde{b}\pi)+\sin\tilde{b}\pi\,\overline{Sin}_n(\tilde{b}\pi)\Big]$$

where by (II.8)

$$\overline{Cos}_n(x)=\sum_{p=n}^{\infty}(-)^p\frac{x^{2p}}{(2p)!}=\cos x-\sum_{p=0}^{n-1}(-)^p\frac{x^{2p}}{(2p)!}$$

$$\overline{Sin}_n(x)=\sum_{p=n}^{\infty}(-)^p\frac{x^{2p+1}}{(2p+1)!}=\sin x-\sum_{p=0}^{n-1}(-)^p\frac{x^{2p+1}}{(2p+1)!}$$

Taking $\gamma=3,\ \tilde{\rho}=15/4,\ \tilde{a}=4,\ \tilde{b}=7/2$, $\boxed{\gamma/2\pi^3 = .0483773}$

$\boxed{\frac{1}{2\tilde{\rho}^2}=.035556}$, $z=\frac{2\pi}{3}$

(terms that enter directly into the sum formula for $\tilde{\zeta}^{(0,1)}$ are boxed in)

$$G(\tfrac{1}{4},z)=\frac{\pi}{2}-4\sin\frac{\pi}{6}+\ln 2-\ln\frac{\pi}{3}-\sum_{n=1}^{\infty}\frac{(2^{2n}-2)}{2n(2n)!}B_n\left(\frac{\pi}{3}\right)^{2n}$$
$$=1.570796-4\times.500000+.693147-.046121-.097722$$
$$=.120100$$

n	$\frac{(2^{2n}-2)B_n}{2n(2n)!}$	$\left(\frac{\pi}{3}\right)^{2n}$
1	.083 333	1.096 622
2	.004 861	1.202 580
3	.000 342	1.318 78
4	.000 026	1.446 23
5	.000 002	1.585 98
6	.000 0002	1.739 23

$\Psi(\tilde{a})=\Psi(4)=\boxed{1.506118}$, $\Psi(\tilde{b})=\Psi(3.5)=\boxed{1.388871}$, $\Psi(\tfrac{1}{4})=\boxed{-.227454}$

$\frac{1}{\tilde{b}}=\boxed{.285714}$, $\cos\tilde{b}z=\cos\frac{7\pi}{3}=\boxed{.500000}$

$$F(\tilde{a},z) = F\left(4, \tfrac{2\pi}{3}\right) = \cos 4\pi\left[-\tfrac{1}{2}\Psi(1)+\tfrac{1}{2}\Psi\left(\tfrac{3}{2}\right)\right] - \tfrac{1}{8}\left[\cos\tfrac{8\pi}{3} + \cos 4\pi\right] - \tfrac{1}{2}\int_{\pi}^{2\pi/3} \cos 4z \cot\tfrac{z}{2}\, dz$$

$$\frac{1}{2}\int_{\pi}^{2\pi/3} \cos 4z \cot\frac{z}{2}\, dz = Ci\left(\frac{8\pi}{3}\right) - Ci(4\pi) + \sum_{n=1}^{\infty} \frac{(-)^n B_n}{2n\,4^{2n}}\left[-\cos\frac{8\pi}{3}\,\overline{Cos}_n\left(\frac{8\pi}{3}\right) - \sin\frac{8\pi}{3}\,\overline{Sin}_n\left(\frac{8\pi}{3}\right) + \cos 4\pi\,\overline{Cos}_n(4\pi) + \sin 4\pi\,\overline{Sin}_n(4\pi)\right]$$

n	$\frac{B_n}{2n\,4^{2n}}$	$(-)^n\overline{Cos}_n\left(\frac{8\pi}{3}\right)$	$(-)^n\overline{Sin}_n\left(\frac{8\pi}{3}\right)$	$(-)^n\overline{Cos}_n(4\pi)$
1	.52083 (-2)	+1.500 000	7.511557	.000 000
2	.32552 (-4)	3.359 194 (1)	9.048384 (1)	7.895 685
3	.96875 (-6)	1.716 490 (2)	2.53400 (2)	9.60074 (2)
4	.63583 (-7)	3.08504 (2)	3.21247 (2)	6.42932 (3)
5	.72250 (-8)	2.93264 (2)	2.38904 (2)	8.99333 (3)
6	.12572 (-8)	1.76007 (2)		1.80672 (4)

(power of ten)

$\cos\frac{8\pi}{3} = -.500000$

$\sin\frac{8\pi}{3} = .866025$

$\cos 4\pi = 1$

$\sin 4\pi = 0$

$$\frac{1}{2}\int_{\pi}^{2\pi/3} \cos 4z \cot\tfrac{z}{2}\, dz = .107433 - (-.006121) + .500000(.009094) - .866025(.042335) + .001690 + 0$$

$$= .083128$$

$$F(\tilde{a},z) = -.500000(.422784) + .500000(.703157) - .125000(-.500000+1) - .083128 = \boxed{-.005442}$$

Similarly

$$F(\hat{b},z) = F\left(\tfrac{7}{2}, \tfrac{2\pi}{3}\right) = \cos\tfrac{7\pi}{2}\left[-\tfrac{1}{2}\Psi\left(\tfrac{3}{4}\right)+\tfrac{1}{2}\Psi\left(\tfrac{5}{4}\right)\right] - \tfrac{1}{7}\left[\cos\tfrac{7\pi}{3} + \cos\tfrac{7\pi}{2}\right] - \tfrac{1}{2}\int_{\pi}^{2\pi/3} \cos\tfrac{7\pi}{2} \cot\tfrac{z}{2}\, dz$$

$$\frac{1}{2}\int_{\pi}^{2\pi/3} \cos 3.5\pi \cot\frac{z}{2}\, dz = Ci\left(\frac{7\pi}{3}\right) - Ci\left(\frac{7\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{B_n (-)^n}{2n(3.5)^{2n}}\left[-\cos\frac{7\pi}{3}\,\overline{Cos}_n\left(\frac{7\pi}{3}\right) - \sin\frac{7\pi}{3}\,\overline{Sin}_n\left(\frac{7\pi}{3}\right) + \cos\frac{7\pi}{2}\,\overline{Cos}_n\left(\frac{7\pi}{2}\right) + \sin\frac{7\pi}{2}\,\overline{Sin}_n\left(\frac{7\pi}{2}\right)\right]$$

$$= .105920 - (-.089564) - .500000(.005094) - .866025(.047537) + 0 - .096312$$

$$= .055457$$

n	$\frac{B_n}{2n(3.5)^{2n}}$	$(-)^n\overline{Cos}_n\left(\frac{7\pi}{3}\right)$	$(-)^n\overline{Sin}_n\left(\frac{7\pi}{3}\right)$	$\overline{Sin}_n\left(\frac{7\pi}{2}\right)$
1	.680272 (-2)	+.500 000	+6.46436	+1.199558 (1)
2	.55532 (-4)	2.636 726(1)	5.918485 (1)	2.095708 (2)
3	.21586 (-5)	9.39410 (1)	1.17196 (2)	1.129826 (3)
4	.18505 (-6)	1.21549 (2)	1.08465 (2)	2.725819 (3)
5	.27464 (-7)	8.522[illegible] (1)	5.9948 (1)	3.748584 (3)
6	.62419 (-8)	3.9229 (1)		

$\cos\frac{7\pi}{3} = .500000$

$\sin\frac{7\pi}{3} = .866025$

$\cos\frac{7\pi}{2} = 0$

$\sin\frac{7\pi}{2} = -1$

Hence

$$F(\tilde{b},\mathfrak{z}) = 0 - .142857(.500000+0) - .055457 = \boxed{-.126886}$$

Thus finally

$$\tilde{\zeta}^{(91)} = .0483773 \times .035555 [-1.506118 - 1.388871 + 2(-.227454) + .285714$$
$$- .005442 + .126886 - .500000(.285714) + 2(.120100)]$$

$$= -.0483773 \times .101171 = -.0048943$$

Direct summation gives for 9 terms $\sum_{n=1}^{9} \tilde{S}_n^{(0,1)} \cong -.005742$. This is smaller than the exact sum, as expected, since the later terms are all positive and still changing in the fourth place. The result also illustrates the difficulty with direct summation, since there is still an error of 4% after using 9 terms, and the convergence is getting slower.

n	$\tilde{S}_n^{(91)}$	$S_n^{(91)}$
1	-.001 548	-.00366
2	-.004 778	-.00541
3	-.002 127	-.00184
4	.001 423	.00160
5	.001 366	.00133
6	.000 155	.00013
7	.000 087	.00009
8	.000 217	.00021
9	.000 037	.00004

APPENDIX IV. Numerical Tables Useful in the Calculation of Z

Table I. (a) Bessel Function Roots and Useful Functions of the Roots

n	β_n	$1/\beta_n^2$	$\dfrac{1}{\beta_n^3 J_0^2(\beta_n)}$	$\dfrac{1}{2\pi^3(n+\frac{1}{2})^3}$
0	0			
1	3.83171	.068111	.109580	.0082564
2	7.01559	.020318	.032153	.0014157
3	10.17347	.009662	.015231	.0004697
4	13.32369	.005633	.008867	.0002101
5	16.47063	.003686	.005798	.0001114
6	19.61586	.002599	.004086	.0000661
7	22.76008	.001930	.003034	.0000423
8	25.90367	.001490	.002343	.0000287
9	29.04683	.001185	.001863	.0000204
10	32.18968	.000965	.001516	.0000150
11	35.33231	.000801	.001259	.0000113
12	38.47477	.000676	.001061	.0000088
13	41.61709	.000577	.0009072	.0000069
14	44.75932	.000499	.0007842	.0000056
15	47.90147		.0006847	
16	51.04354		.0006030	
17	54.18555		.0005351	
18	57.32752		.0004780	
19	60.46946		.0004296	
20	63.61136		.0003882	
21	66.75323		.0003525	
22	69.89507		.0003216	
23	73.03690		.0002945	
24	76.17870		.0002707	
25	79.32049		.0002497	

APPENDIX IV.

Table I. (b) Coefficients in Series Expansions of $\tilde{S}^{(a,0)}(\gamma)$

n	$\frac{(2^{2n}-2)B_n}{2n(2n+2)!}$	$\frac{E_n}{(2n+3)!}$
0		.166 666 667
1	.006 944 444	.008 333 333
2	.000 162 037	.000 992 064
3	.000 006 102	.000 168 100
4	.000 000 292	.000 034 697
5	.000 000 016	.000 008 113
6	.000 000 001	.000 002 067
7		.000 000 561

Table II (a) Values of Basic Sum $S^{(0,0)}(\gamma)$ and its Asymptotic Series $\tilde{S}^{(0,0)}(\gamma)$ for Various γ

γ	$S^{(0,0)}(\gamma)$	$\tilde{S}^{(0,0)}(\gamma)$
1	0	0
1.2	.012355	.011947
1.4	.02463	.02288
1.6	.03269	.02884
2.0	.04018	.03155
2.5	.04193	.02809
3.0	.040622	.023184
4.0	.03594	.01566
5.0	.03146	.01238
6.0	.02769	.01244
8.0	.02222	.01904
10.0	.01845	.03075

$$S_n^{(0,0)} = \frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \; ; \quad \tilde{S}_n^{(0,0)} = -\frac{\gamma}{2\pi^3}\,\frac{\left(1-\sin(n+1/4)\,2\pi/\gamma\right)}{(n+1/4)^3}$$

$$S^{(0,0)}(\gamma) = \sum_{n=1}^{\infty} S_n^{(0,0)}(\gamma) \; ; \quad \tilde{S}^{(0,0)}(\gamma) = \sum_{n=1}^{\infty} \tilde{S}_n^{(0,0)}(\gamma)$$

Table II (b) Term by Term Values of Basic Sums $S^{(0,0)}(\gamma)$ and $\tilde{S}^{(0,0)}(\gamma)$ with partial and total sums

n	$S_n^{(0,0)}(1.2)$	$\tilde{S}_n^{(0,0)}(1.2)$	$S_n^{(0,0)}(1.4)$	$\tilde{S}_n^{(0,0)}(1.4)$	$S_n^{(0,0)}(1.6)$	$\tilde{S}_n^{(0,0)}(1.6)$
1	.007644	.007343	.02027	.01877	.02979	.02617
2	.002972	.002900	.00348	.00322	.00126	.00101
3	.001134	.001108	.00008	.00007	.00057	.00061
4	.000326	.000317	.00022	.00023	.00063	.00062
5	.000041	.000039	.00031	.00031	.00001	.00000
6	.000002	.000003	.00007	.00007	.00016	.00016
7	.000037	.000038	.00001	.00001	.00008	.00008
8	.000059	.000059	.00006	.00007	.00001	.00001
9	.000048	.000048	.00005	.00005	.00006	.00006
	$\sum_1^9 = .012213$ $S = .012305$	$\sum_1^9 = .011855$ $\tilde{S} = .011947$	$\sum_1^9 = .02455$ $S = .02463$	$\sum_1^9 = .02280$ $\tilde{S} = .02288$	$\sum_1^9 = .03257$ $S = .03269$	$\sum_1^9 = .02872$ $\tilde{S} = .02884$

Table II(b) (cont.)

n	$S_n^{(00)}(2)$	$\tilde{S}_n^{(00)}(2)$	$S_n^{(00)}(2.5)$	$\tilde{S}_n^{(00)}(2.5)$	$S_n^{(00)}(3)$	$\tilde{S}_n^{(00)}(3)$	$S_n^{(00)}(4)$	$\tilde{S}_n^{(00)}(4)$	$S_n^{(00)}(5)$	$\tilde{S}_n^{(00)}(5)$
0										
1	.03695	.02819	.03465	.02064	.029284	.012385	.01989	.00251	.01387	.00000
2	.00058	.00083	.00534	.00562	.009131	.008494	.01083	.00783	.00946	.00489
3	.00172	.00160	.00013	.00006	.000510	.000705	.00360	.00362	.00502	.00425
4	.00010	.00012	.00106	.00102	.000409	.000315	.00038	.00052	.00181	.00190
5	.00040	.00038	.00010	.00011	.000678	.000668	.00007	.00003	.00029	.00038
6	.00003	.00004	.00018	.00017	.000083	.000099	.00041	.00037	.00001	.00000
7	.00015	.00014	.00016	.00017	.000074	.000063	.00032	.00033	.00018	.00015
8	.00002	.00002	.00001	.00000	.000173	.000172	.00006	.00007	.00027	.00026
9	.00007	.00007	.00010	.00010	.000027	.000031	.00001	.00001	.00018	.00018
10					.000025	.000023				
11					.000068	.000068				
12					.000012	.000013				
13					.000011	.000010				
14					.000033	.000034				
	$\sum_1^9 = .04002$	$\sum_1^9 = .02139$	$\sum_1^9 = .04173$	$\sum_1^9 = .02789$	$\sum_1^{14} = .040518$	$\sum_1^{14} = .023080$	$\sum_1^9 = .03557$	$\sum_1^9 = .01529$	$\sum_1^9 = .03109$	$\sum_1^9 = .01201$
	$S = .04018$	$\tilde{S} = .03135$	$S = .04193$	$\tilde{S} = .02809$	$S = .040622$	$\tilde{S} = .023184$	$S = .03594$	$\tilde{S} = .01566$	$S = .03145$	$\tilde{S} = .01238$

Table II(b)(cont.)

n	$S_n^{(0,0)}(6)$	$\tilde{S}_n^{(0,0)}(6)$	$S_n^{(0,0)}(8)$	$\tilde{S}_n^{(0,0)}(8)$	$S_n^{(0,0)}(10)$	$\tilde{S}_n^{(0,0)}(10)$
0						
1	.01008	.00169	.00593	.01113	.00388	.02418
2	.00773	.00249	.00508	.00022	.00349	.00017
3	.00508	.00355	.00405	.00167	.00302	.00051
4	.00271	.00248	.00294	.00201	.00248	.00115
5	.00106	.00114	.00190	.00163	.00191	.00129
6	.00022	.00029	.00106	.00105	.00137	.00113
7	.00000	.00001	.00047	.00053	.00090	.00084
8	.00007	.00005	.00014	.00018	.00053	.00054
9	.00017	.00015	.00001	.00003	.00026	.00030
10	.00018	.00018	.00001	.00000	.00010	.00013
11	.00011	.00012	.00005	.00004	.00002	.00003
12	.00003	.00004	.00010	.00008	.00000	.00000
13	.00000	.00000	.00011	.00010	.00001	.00001
14	.00001	.00001	.00009	.00009	.00004	.00003
	$\sum_1^{14}=.02745$	$\sum_1^{14}=.01220$	$\sum_1^{14}=.02194$	$\sum_1^{14}=.01876$	$\sum_1^{14}=.01801$	$\sum_1^{14}=.03031$
	$S=.02769$	$\tilde{S}=.01244$	$S=.02222$	$\tilde{S}=.01904$	$S=.01845$	$\tilde{S}=.03075$

Table III The Complex Series $\mathcal{L}^{(0,0)}(\gamma, \varkappa'R')$ Entering the First Variational Term for Z, for Various γ and $\varkappa'R'$.

$$\mathcal{L}^{(0,0)}(\gamma, \varkappa'R') = \sum_{n=1}^{\infty} \frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \cdot \frac{1}{\left[1-\left(\frac{\varkappa'R'}{\beta_n}\right)^2\right]^{1/2}}$$

$(\varkappa'R')^2$	$\varkappa'R'$	$\gamma = 1.2$ $\mathcal{L}^{(0,0)}(1.2, \varkappa'R')$	$\gamma = 1.4$ $\mathcal{L}^{(0,0)}(1.4, \varkappa'R')$	$\gamma = 1.6$ $\mathcal{L}^{(0,0)}(1.6, \varkappa'R')$
0	0	.01236	.02463	.03269
.001i	.02236(1+i)	.01236 + 0 i	.02463 + .000001i	.03269 + .000001i
.005i	.05000(1+i)	.01236 + .000002i	.02463 + .000003i	.03269 + .000005i
.02i	.10000(1+i)	.01236 + .000006i	.02463 + .000015i	.03269 + .000020i
.05i	.15811(1+i)	.01236 + .000015i	.02463 + .000037i	.03269 + .000052i
0.1i	.22361(1+i)	.01236 + .000030i	.02463 + .000073i	.03269 + .000102i
0.2i	.31623(1+i)	.01236 + .000059i	.02463 + .000145i	.03269 + .000207i
0.5i	.50000(1+i)	.01236 + .000148i	.02462 + .000363i	.03268 + .000515i
1.0i	.70711(1+i)	.01234 + .000296i	.02459 + .000724i	.03264 + .00102i
2.0i	1.00000(1+i)	.01230 + .000588i	.02449 + .00143i	.03249 + .00205i
5.0i	1.58114(1+i)	.01204 + .00140i	.02381 + .00340i	.03149 + .00482i

Table III
(cont.)

$$\mathcal{S}^{(0,0)}(\gamma, R'\mathcal{H}') = \sum_{n=1}^{\infty} \frac{J_1^2(\beta_n/\gamma)}{\beta_n^3 J_0^2(\beta_n)} \cdot \frac{1}{\left[1-\left(\frac{\mathcal{H}'R'}{\beta_n}\right)^2\right]^{1/2}}$$

$(\mathcal{H}'R')^2$	$\mathcal{H}'R'$	$\gamma = 2.0$ $\mathcal{S}^{(0,0)}(2.0, \mathcal{H}'R')$	$\gamma = 2.5$ $\mathcal{S}^{(0,0)}(2.5, \mathcal{H}'R')$
0	0	.04018	.04193
.001i	.02236(1+i)	.04018 + .000001i	.04193 + .000001i
.005i	.05000(1+i)	.04018 + .000006	.04193 + .000006i
.02i	.00000(1+i)	.04018 + .000025	.04193 + .000025i
.05i	.15811(1+i)	.04018 + .000063	.04193 + .000062i
0.1i	.22361(1+i)	.04018 + .000128	.04193 + .000123i
0.2i	.31623(1+i)	.04018 + .000255	.04193 + .000248i
0.5i	.50000(1+i)	.04016 + .000636	.04192 + .000618i
1.0i	.70711(1+i)	.04010 + .00127	.04187 + .00123i
2.0i	1.00000(1+i)	.03993 + .00252	.04169 + .00245i
5.0i	1.58114(1+i)	.03870 + .00594	.04052 + .00579i

Table III (cont.)

$\mathcal{S}^{(0,0)}(\gamma, \varkappa' R')$

$(\varkappa' R')^2$	$\gamma = 3.0$ $\mathcal{S}^{(0,0)}(3.0, \varkappa' R')$	$\gamma = 4.0$ $\mathcal{S}^{(0,0)}(4.0, \varkappa' R')$	$\gamma = 5.0$ $\mathcal{S}^{(0,0)}(5.0, \varkappa' R')$
0	.04062	.03594	.03146
.001i	.04062 + .000001i	.03594 + .000001i	.03146
.005i	.04062 + .000005i	.03594 + .000004i	.03146 + .000002i
.02i	.04062 + .000022i	.03594 + .000016i	.03146 + .000011i
.05i	.04062 + .000055i	.03594 + .000041i	.03146 + .000030i
0.1i	.04062 + .000109i	.03594 + .000081i	.03146 + .000060i
0.2i	.04062 + .000218i	.03594 + .000160i	.03146 + .000119i
0.5i	.04061 + .000547i	.03593 + .000403i	.03145 + .000299i
1.0i	.04057 + .001091i	.03590 + .000803i	.03144 + .000597i
2.0i	.04041 + .00216i	.03577 + .00159i	.03135 + .001186i
5.0i	.03941 + .00514i	.03510 + .003[illegible]i	.03086 + .00283i

Table III (cont.)

$\mathcal{L}^{(0,0)}(\gamma, \mathcal{K}'R')$

$(\mathcal{K}'R')^2$	$\gamma = 6.0$ $\mathcal{L}^{(0,0)}(6.0, \mathcal{K}'R')$	$\gamma = 8.0$ $\mathcal{L}^{(0,0)}(8.0, \mathcal{K}'R')$	$\gamma = 10.0$ $\mathcal{L}^{(0,0)}(10.0, \mathcal{K}'R')$
0	.02769	.02222	.01845
.001i	.02769	.02222	.01845
.005i	.02769 + .000002i	.02222 + .000001i	.01845 + .000001i
.02i	.02769 + .000009i	.02222 + .000005i	.01845 + .000004i
.05i	.02769 + .000022i	.02222 + .000014i	.01845 + .000010i
0.1i	.02769 + .000045i	.02222 + .000028i	.01845 + .000018i
0.2i	.02769 + .000092i	.02222 + .000057i	.01845 + .000038i
0.5i	.02769 + .000228i	.02222 + .000144i	.01845 + .000097i
1.0i	.02767 + .000456i	.02221 + .000286i	.01844 + .000196i
2.0i	.02762 + .000905i	.02218 + .000569i	.01842 + .00038i
5.0i	.02726 + .00216i	.02196 + .001366i	.01828 + .00093i

Table IV. First Variational Approximation to Series Impedance for Change of Cross Section (No change of material) $\frac{Z}{Z_0} \cong -4i\kappa'R'\mathcal{S}^{(90)}(\gamma,\kappa'R')$

and to Reflection Coefficient $\mathcal{R}$ (Semi-infinite Termination)

(θ is in radians and is in the 3rd quadrant i.e. $-\pi \le \theta \le -\frac{\pi}{2}$)

$\mathcal{R} = |\mathcal{R}|e^{i\theta} = \frac{t-1}{t+1}$; $t = 1/\gamma^2 + Z/Z_0$

$(\kappa'R')^2$	$\gamma=1.2$ Z/Z_0	$\gamma=1.2$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=1.4$ Z/Z_0	$\gamma=1.4$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=1.6$ Z/Z_0
0	0 0	.1803	.0000	0 0	.3243	0	0 0
.001i	.0011 − .0011i	.1796	.0043	.0022 − .0022i	.3224	.0060	.0029 − .0029i
.005i	.0025 − .0025i	.1786	.0096	.0049 − .0049i	.3200	.0134	.0065 − .0065i
.02i	.0049 − .0049i	.1769	.0193	.0099 − .0098i	.3159	.0270	.0131 − .0131i
.05i	.0078 − .0078i	.1750	.0308	.0156 − .0155i	.3109	.0430	.0207 − .0206i
0.1i	.0111 − .0110i	.1728	.0439	.0221 − .0220i	.3056	.0612	.0293 − .0291i
0.2i	.0157 − .0156i	.1697	.0627	.0313 − .0310i	.2980	.0875	.0416 − .0411i
0.5i	.0250 − .0244i	.1637	.1011	.0500 − .0485i	.2835	.1410	.0664 − .0643i
1.0i	.0357 − .0341i	.1572	.1453	.0716 − .0675i	.2676	.2027	.0952 − .0894i
2.0i	.0516 − .0468i	.1479	.2091	.1037 − .0922i	.2456	.2917	.1381 − .1217i
5.0i	.0850 − .0673i	.1295	.3340	.1721 − .1291i	.2032	.4624	.2296 − .1686i

$(\kappa'R')^2$	$\gamma=2.0$ Z/Z_0	$\gamma=2.0$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=2.5$ Z/Z_0	$\gamma=2.5$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=3.0$ Z/Z_0
0	0	.6000	0	0	.7241	0	0
.001i	.0036 − .0036i	.5954	.0077	.0038 − .0038i	.7186	.0077	.0036 − .0036i
.005i	.0080 − .0080i	.5898	.0172	.0084 − .0084i	.7118	.0173	.0081 − .0081i
.02i	.0161 − .0161i	.5798	.0346	.0168 − .0168i	.6996	.0346	.0162 − .0162i
.05i	.0254 − .0254i	.5684	.0548	.0266 − .0265i	.6857	.0548	.0257 − .0256i
0.1i	.0360 − .0358i	.5557	.0779	.0376 − .0374i	.6704	.0779	.0364 − .0362i
0.2i	.0511 − .0505i	.5381	.1109	.0533 − .0527i	.6491	.1103	.0516 − .0511i
0.5i	.0816 − .0790i	.5046	.1769	.0850 − .0826i	.6086	.1752	.0823 − .0801i
1.0i	.1170 − .1098i	.4685	.2520	.1219 − .1149i	.5651	.2482	.1178 − .1116i
2.0i	.1698 − .1496i	.4197	.3573	.1765 − .1569i	.5067	.3491	.1702 − .1530i
5.0i	.2823 − .2071i	.3308	.5514	.2928 − .2196i	.4013	.5315	.2817 − .2167i

$(\kappa'R')^2$	$\gamma=4$ Z/Z_0	$\gamma=4$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=5$ Z/Z_0	$\gamma=5$ $\lvert\mathcal{R}\rvert$	$\theta+\pi$	$\gamma=6$ Z/Z_0
0	0	.8824	0	0	.9231	0	0
.001i	.0032 − .0032i	.8767	.0065	.0028 − .0028i	.9179	.0056	.0025 − .0025i
.005i	.0072 − .0072i	.8697	.0145	.0063 − .0063i	.9116	.0126	.0055 − .0055i
.02i	.0144 − .0144i	.8573	.0289	.0126 − .0126i	.9003	.0252	.0111 − .0111i
.05i	.0228 − .0227i	.8430	.0458	.0199 − .0199i	.8870	.0399	.0175 − .0175i
0.1i	.0322 − .0321i	.8271	.0647	.0282 − .0281i	.8724	.0564	.0248 − .0247i
0.2i	.0457 − .0453i	.8052	.0915	.0399 − .0396i	.8522	.0797	.0351 − .0349i
0.5i	.0727 − .0711i	.7629	.1446	.0635 − .0623i	.8131	.1257	.0558 − .0549i
1.0i	.1038 − .0993i	.7187	.2134	.0906 − .0872i	.7705	.1769	.0796 − .0770i
2.0i	.1496 − .1368i	.6557	.2844	.1301 − .1206i	.7129	.2470	.1141 − .1069i
5.0i	.2462 − .1940i	.5434	.4292	.2130 − .1772i	.6067	.3733	.1861 − .1588i

(Values of $\lvert\mathcal{R}\rvert$ and $\theta+\pi$ for $\gamma = 6$ are on following page.)

Table IV (cont.) First Variational Approximation to Series Impedance for Change of Cross Section (No change of material)

and to Reflection Coefficient $\mathcal{R}$ (Semi-infinite Termination)

(θ is in radians and is in the 3rd quadrant i.e. $-\pi \leq \theta \leq -\frac{\pi}{2}$)

$(\mathcal{X}'R')^2$	$\gamma = 1.6$ $\lvert\mathcal{R}\rvert$	$\gamma = 1.6$ $\theta+\pi$	$\gamma = 8$ Z/Z_0	$\gamma = 8$ $\lvert\mathcal{R}\rvert$	$\gamma = 8$ $\theta+\pi$	$\gamma = 10$ Z/Z_0	$\gamma = 10$ $\lvert\mathcal{R}\rvert$	$\gamma = 10$ $\theta+\pi$
0	.4382	0	0	.9692	0	0	.9802	0
.001i	.4352	.0069	.0020 − .0020i	.9654	.0040	.0016 − .0016i	.9770	.0033
.005i	.4315	.0155	.0044 − .0044i	.9606	.0089	.0037 − .0037i	.9730	.0074
.02i	.4249	.0312	.0089 − .0089i	.9522	.0178	.0074 − .0074i	.9658	.0148
.05i	.4173	.0497	.0141 − .0140i	.9424	.0281	.0117 − .0117i	.9576	.0233
0.1i	.4069	.0707	.0199 − .0198i	.9314	.0398	.0165 − .0165i	.9483	.0330
0.2i	.3973	.1009	.0282 − .0280i	.9162	.0561	.0234 − .0233i	.9354	.0466
0.5i	.3749	.1620	.0447 − .0442i	.8864	.0886	.0371 − .0367i	.9102	.0736
1.0i	.3506	.2324	.0636 − .0620i	.8536	.1247	.0527 − .0516i	.8823	.1035
2.0i	.3174	.3322	.0910 − .0864i	.8086	.1744	.0752 − .0722i	.8438	.1451
5.0i	.2551	.5215	.1475 − .1302i	.7233	.2660	.1215 − .1097i	.7701	.2224

$(\mathcal{X}'R')^2$	$\gamma = 3.0$ $\lvert\mathcal{R}\rvert$	$\gamma = 3.0$ $\theta+\pi$
0	.6000	0
.001i	.7941	.0074
.005i	.7869	.0165
.02i	.7741	.0330
.05i	.7594	.0523
0.1i	.7431	.0741
0.2i	.7207	.1048
0.5i	.6777	.1660
1.0i	.6314	.2343
2.0i	.5694	.3285
5.0i	.4574	.4972

$(\mathcal{X}'R')^2$	$\gamma = 6$ $\lvert\mathcal{R}\rvert$	$\gamma = 6$ $\theta+\pi$
0	.9459	0
.001i	.9412	.0050
.005i	.9356	.0111
.02i	.9252	.0222
.05i	.9134	.0351
0.1i	.9001	.0496
0.2i	.8818	.0701
0.5i	.8461	.1105
1.0i	.8072	.1553
2.0i	.7540	.2172
5.0i	.6551	.3293

Table V (a) Higher Order Series $\mathcal{S}^{(0,1)}$, $\mathcal{S}^{(1,1)}$ at selected γ and $\kappa' R'$
(with $\mathcal{S}^{(0,0)}$ repeated)

$(\kappa' R')^2$	$\mathcal{S}^{(0,0)}$	$\mathcal{S}^{(0,1)}$	$\mathcal{S}^{(1,1)}$	γ
.001i	.01236	−.01008	.05111	1.2
0.1i	.01236 + .00003i	−.01008 − .00005i	.05111 + .00014i	
5.0i	.01204 + .00140i	−.00940 − .00246i	.04949 + .00679i	
.001i	.04061	−.00728	.02009	3.0
0.1i	.04061 + .00011i	−.00728 − .00002i	.02009	
5.0i	.03940 .00514i	−.00711 − .00086i	.02005 + .00050i	
.001i	.01845	−.00222	.00599	10.0
0.1i	.01845 + .00002i	−.00222	.00599	
5.0i	.01828 + .00093i	−.00222 − .00003i	.00599	

Table V (b) Values of Z/Z_0 and R to Second Variational Approximation at these selected γ and $\kappa'R'$ values (with first approximation repeated)

$(\kappa'R')^2$	γ	Z/Z_0	$\lvert R\rvert$	$\theta+\pi$
.001i	1.2	A .00111 − .00111i	.1796	.0043
		B .00102 − .00102i	.1796	.0039
0.1i	1.2	A .01109 − .01103i	.1728	.0439
		B .01016 − .01019i	.1734	.0404
5.0i	1.2	A .08501 − .06729i	.1295	.3340
		B .07615 − .05779i	.1336	.2793
.001i	3.0	A .00363 − .00363i	.7941	.0074
		B .00352 − .0035[illegible]i	.7943	.0073
0.1i	3.0	A .03642 − .03622i	.7431	.0741
		B .03529 − .03510i	.7449	.0717
5.0i	3.0	A .28170 − .21668i	.4574	.4972
		B .27244 − .20717i	.4649	.4728
.001i	10.0	A .00166 − .00166i	.9770	.0033
		B .00162 − .00162i	.9770	.0032
0.1i	10.0	A .01657 − .01654i	.94[illegible]3	.0330
		B .01622 − .01619i	.9469	.0324
5.0i	10.0	A .12149 − .10973i	.7701	.2224
		B .11893 − .107[illegible]7i	.7740	.21[illegible]

(A = First approximation B = Second approximation)